第２回統数研/総研大経済学研究会

Institute of Statistical Mathematics, Tokyo, 11.11.2002

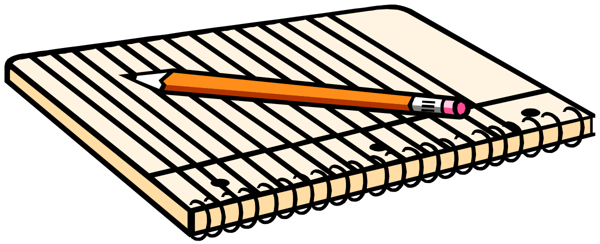
Everything you always wanted to know about the Levy-stable law, but were afraid to ask

Rafał Weron

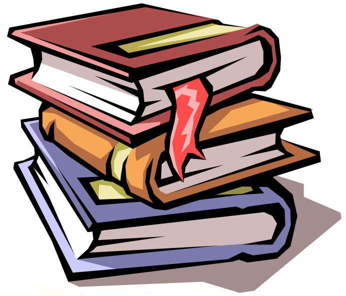
HCS

Topics

- **Introduction**
  - **The STABLE books**
  - **A bit of history**
- Properties of stable laws
- Computer simulation of stable variables
- Estimation of parameters
- Other interesting topics



The STABLE books



Gnedenko-Kolmogorov (1954)

ОДНОМЕРНЫЕ УСТОЙЧИВЫЕ РАСПРЕДЕЛЕНИЯ

Владимир Золотарев

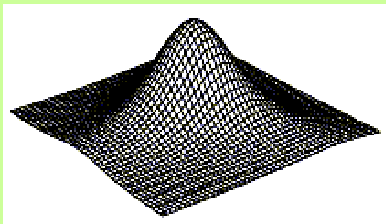
(1983, 1986)

PURE AND APPLIED MATHEMATICS
A Series of Monographs and Textbooks

SIMULATION AND CHAOTIC BEHAVIOR OF α-STABLE STOCHASTIC PROCESSES

Aleksander Janicki
Aleksander Weron

(1994)



Stable Distributions
Models for Heavy Tailed Data

John P. Nolan

(2003)

MODERN PROBABILITY AND STATISTICS

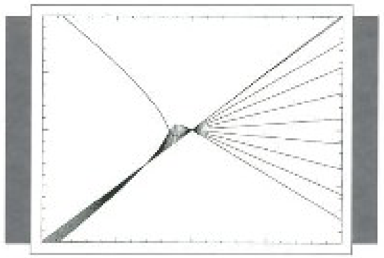
Chance and Stability
Stable Distributions and Their Applications

Vladimir V. UCHAIKIN and Vladimir M. ZOLOTAREV

///VSP///

(1999)

STABLE NON-GAUSSIAN RANDOM PROCESSES
Stochastic Models with Infinite Variance



Gennady Samorodnitsky
Murad S. Taqqu

(1994)

A bit of history

- Cauchy (~1850) extended the theory of errors, generalizing the Gaussian formula to

$$f_N(x) = \frac{1}{\pi} \int_0^\infty \exp(-ct^N) \cos(tx) \, dt$$

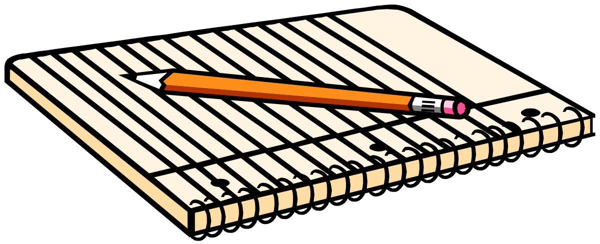
  he succeeded in evaluating the integral only for $N=1$
- Bernstein (1919) observed that f_N is positive definite (and hence a pdf) only when $0 < N \leq 2$
- **Paul Levy (1924)** studied sums of independent variables; found a generalization of the CLT
  - i.i.d. + finite variance ⇒ Gaussian
  - **Levy: i.i.d. ⇒ stable**
  - Levy-Khinchin: i.d. ⇒ infinitely-divisible (eg. NIG, hyperbolic laws)

A bit of applications history

- Holtsmark (1915) – gravitational field of stars (3/2-stable)
- ***Finance***
  - Mandelbrot (1963), Fama (1965), Samuelson (1967), Roll (1970), Akgiray-Booth (1989), Mantegna-Stanley (1995), McCulloch (1996), Cont-Potters-Bouchaud (1997), **Rachev-Mittnik (2000)**
- ***Signal processing***
  - Stuck-Kleiner (1974), Zolotarev (1986), **Nikias-Shao (1995)**, Kuruoglu (1998), Tsakalides-Nikias (1998)
- ***Statistical physics*** *(⇒Paretian/power-law tails)*
  - Jona-Lasinio (1975), Scher-Montrol (1975), Montrol-Shlesinger (1982), West-Seshadri (1982), Takayasu (1984), Tsallis (1997)

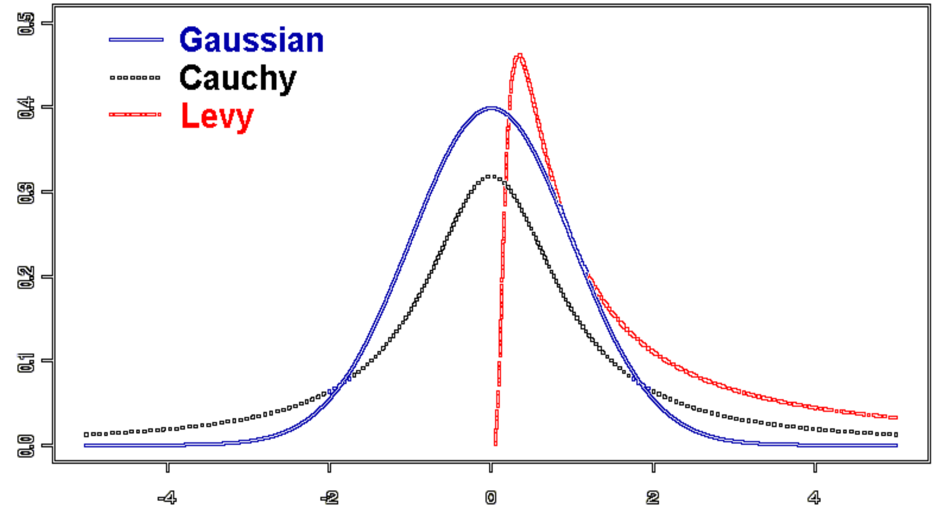
Topics

- Introduction
- **Properties of stable laws**
  - **General properties**
  - **Stability under summation**
  - **Characteristic function representations**
  - **Power-law tails**
- Computer simulation of stable variables
- Estimation of parameters
- Other interesting topics



General properties

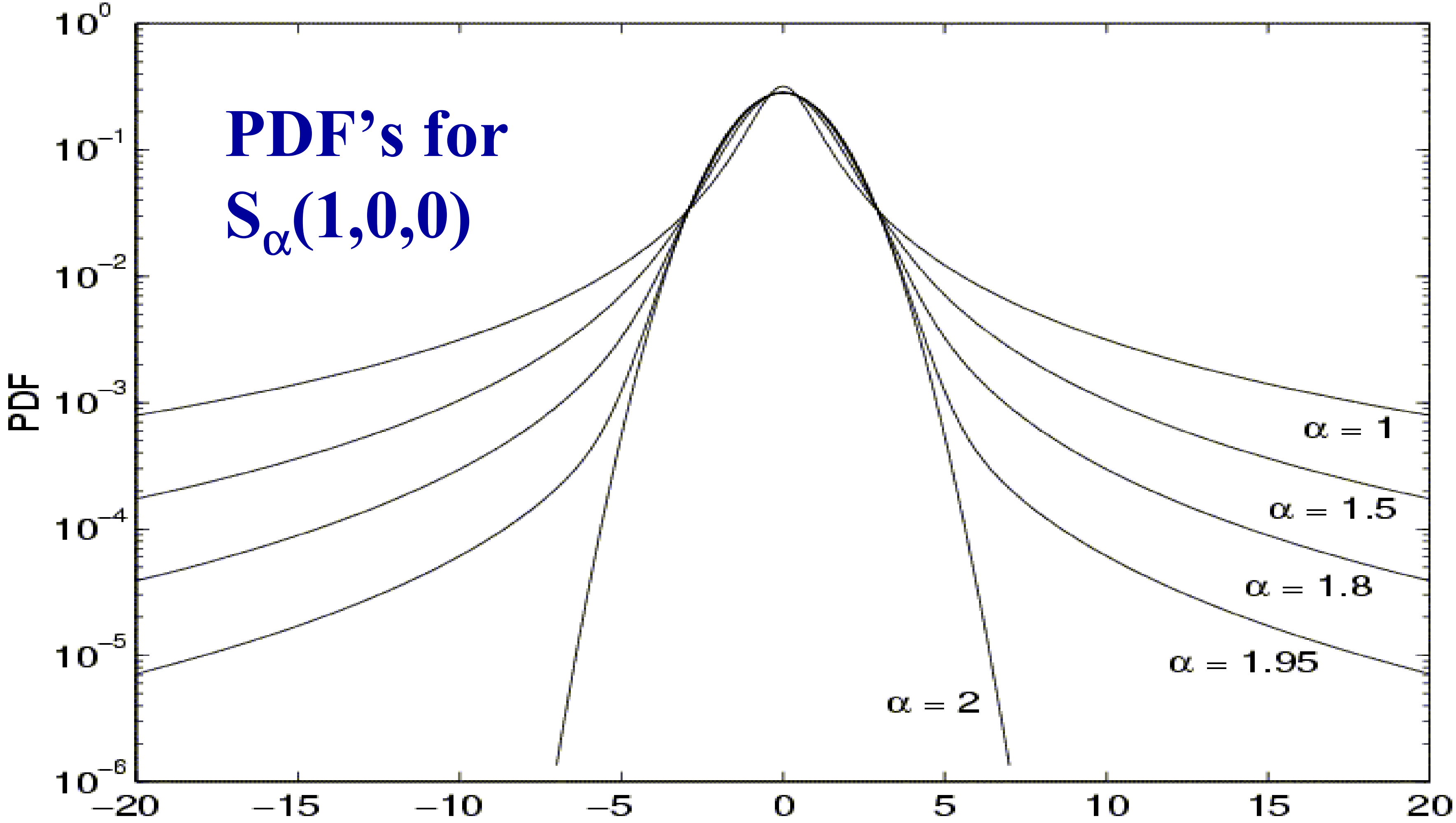
- A four parameter family: $S_\alpha(\sigma,\beta,\mu)$
- PDF's in "closed form" for
  - $\alpha=2$ (Gaussian, normal)
  - $\alpha=1$ (Cauchy)
  - $\alpha=0.5$ (Levy, $|\beta|=1$)



- For $\alpha<2$ variance is infinite ($EX^P<\infty$ only for $p<\alpha$)
- σ - scale parameter
- μ - location parameter; $\mu=EX$ for $\alpha>1$

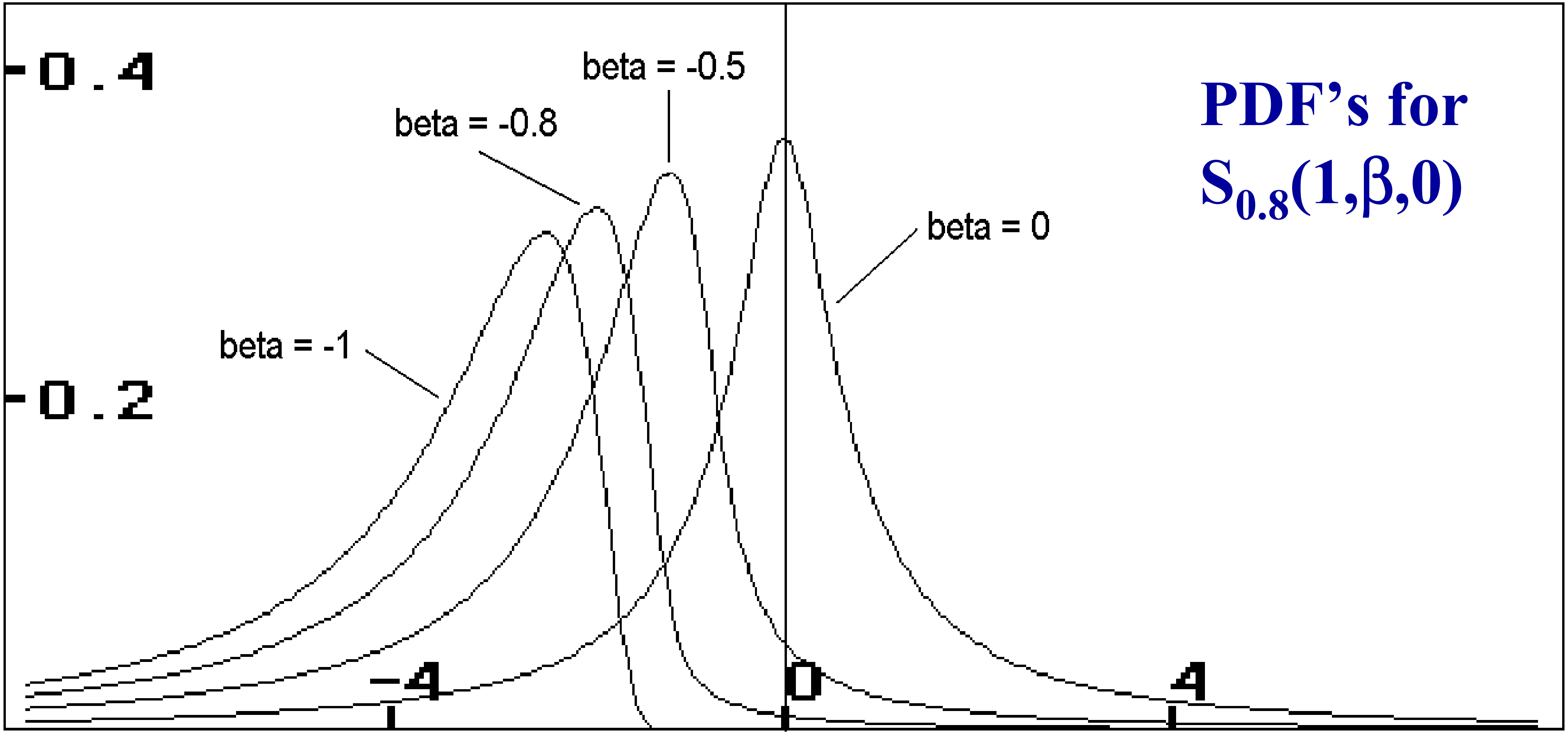
General properties: α

- Index of stability **$\alpha \in (0,2]$** (tail index, tail/characteristic exponent)

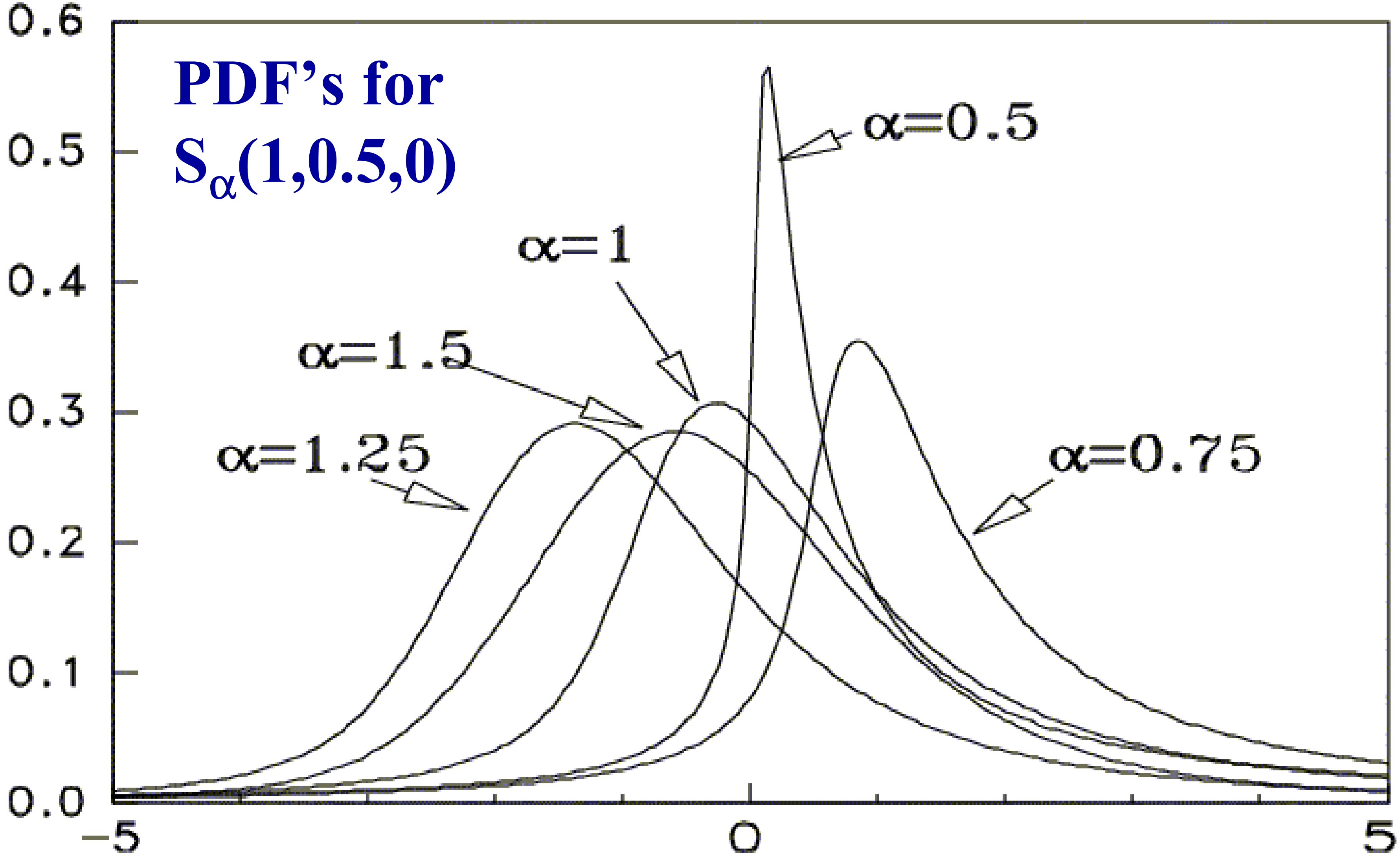


General properties: β

- Skewness parameter $\beta \in [-1,1]$



General properties: α and β



Stability under summation

Proposition 2.1 *If* $X_i \sim S_\alpha(\sigma_i, \beta_i, \mu_i)$ *for* $i = 1, 2$ *are independent random variables, then*

$$X_1 + X_2 \sim S_\alpha(\sigma, \beta, \mu),$$

with

$$\sigma = (\sigma_1^\alpha + \sigma_2^\alpha)^{1/\alpha}, \qquad \beta = \frac{\beta_1\sigma_1^\alpha + \beta_2\sigma_2^\alpha}{\sigma_1^\alpha + \sigma_2^\alpha}, \qquad \mu = \mu_1 + \mu_2.$$

- Summation scheme holds true only for X_i's with **the same α's**
- Otherwise, the sum is infinitely divisible but not stable

Characteristic function representations

- Standard representation $S_\alpha(\sigma,\beta,\mu)$

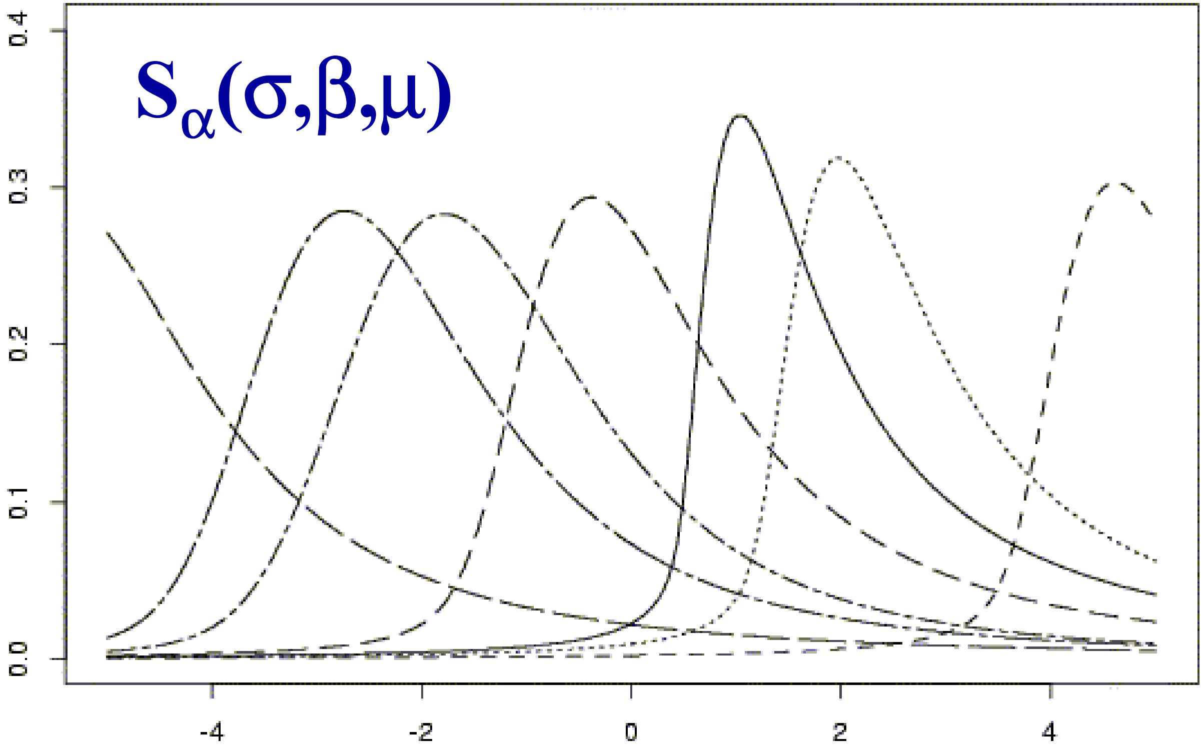
$$\log\phi(t) = \begin{cases} -\sigma^\alpha|t|^\alpha\{1 - i\beta\,\mathrm{sign}(t)\tan\frac{\pi\alpha}{2}\} + i\mu t, & \alpha\neq 1,\\ -\sigma|t|\{1 + i\beta\,\mathrm{sign}(t)\frac{2}{\pi}\log|t|\} + i\mu t, & \alpha = 1.\end{cases}$$

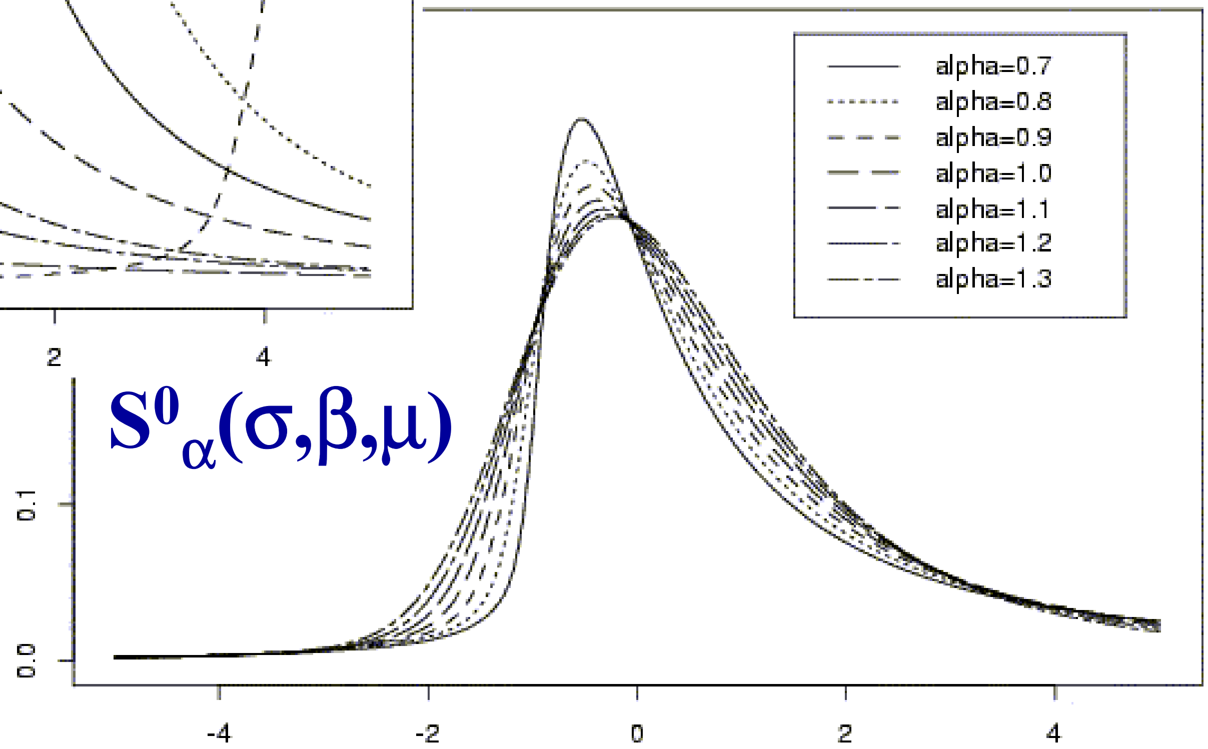
- Zolotarev's M representation $S^0_\alpha(\sigma,\beta,\mu_0)$

$$\log\phi_0(t) = \begin{cases} -\sigma^\alpha|t|^\alpha\{1 + i\beta\,\mathrm{sign}(t)\tan\frac{\pi\alpha}{2}[(\sigma|t|)^{1-\alpha} - 1]\} + i\mu_0 t, & \alpha\neq 1,\\ -\sigma|t|\{1 + i\beta\,\mathrm{sign}(t)\frac{2}{\pi}\log(\sigma|t|)\} + i\mu_0 t, & \alpha = 1.\end{cases}$$

  - PDF is continuous in all 4 parameters, σ and μ_0 are scale and location parameters, i.e. $\sigma X+\mu_0 \sim S^0_\alpha(\sigma,\beta,\mu_0)$ for $X\sim S^0_\alpha(1,\beta,0)$

Characteristic function representations cont.





Power-law tails: Theory

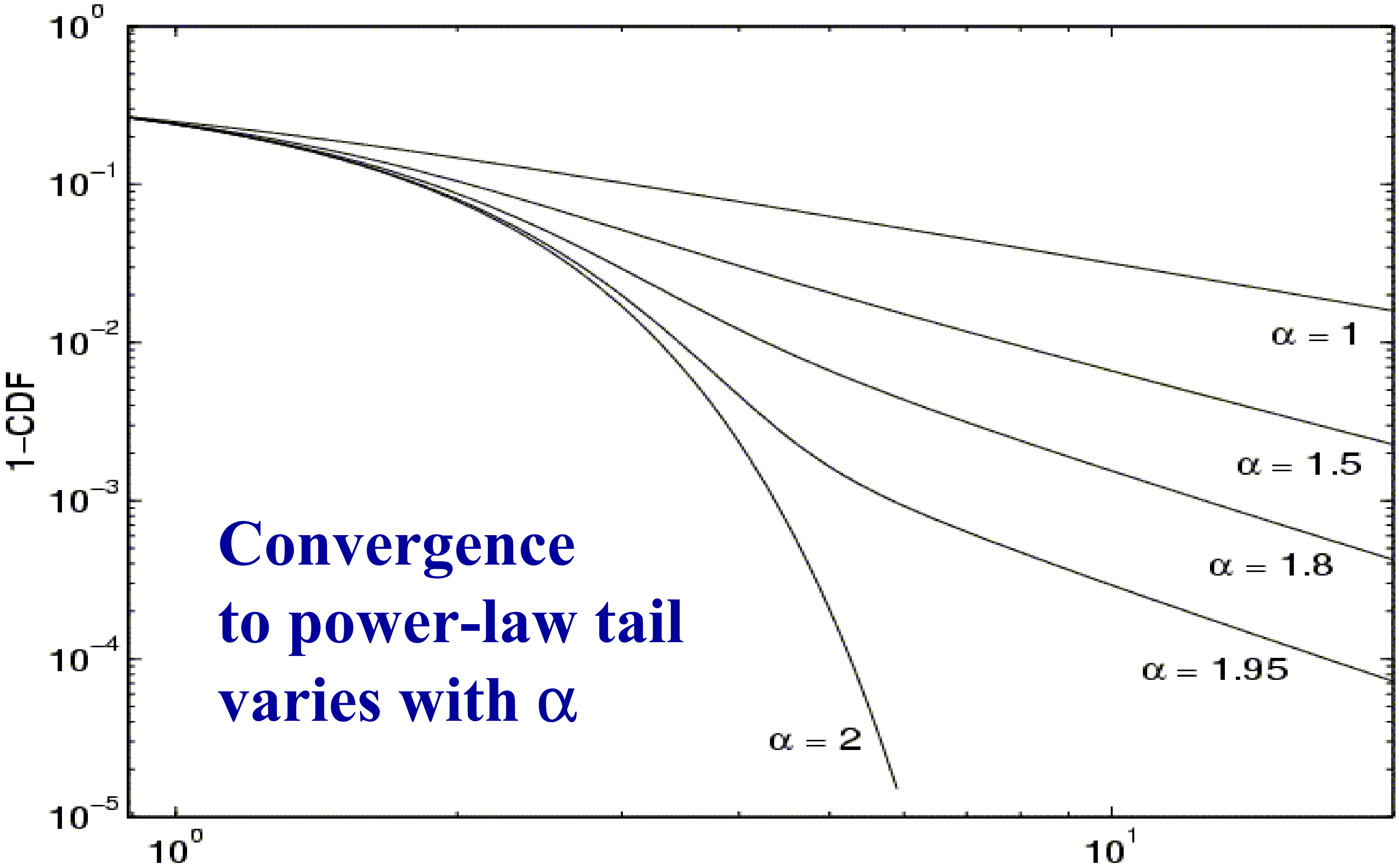
- If $X \sim S_{\alpha<2}(1,\beta,0)$ then

$$P(X > x) = 1 - F(x) \quad \to \quad C_\alpha (1+\beta) x^{-\alpha},$$

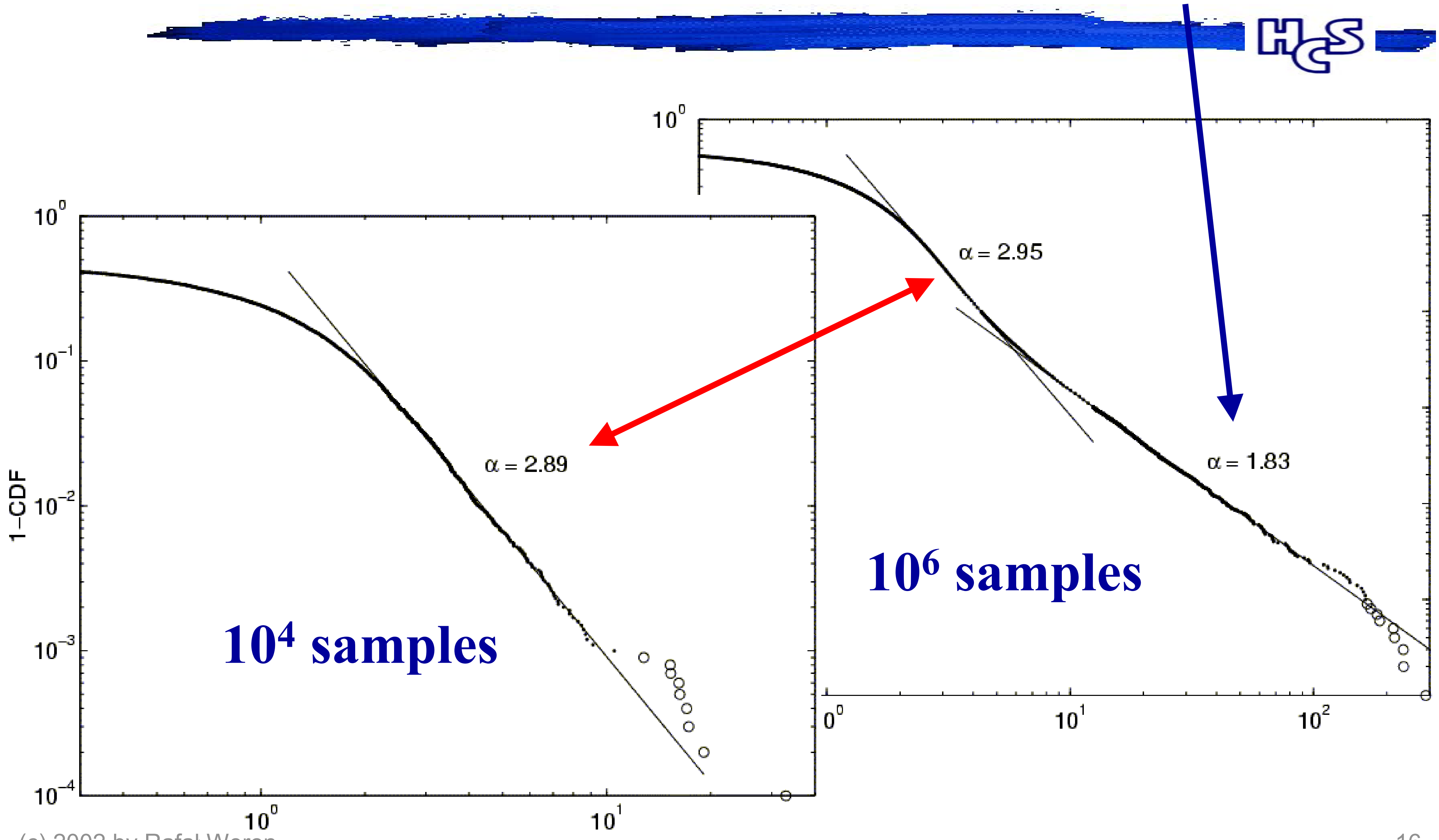
$$P(X < -x) = F(-x) \quad \to \quad C_\alpha (1-\beta) x^{-\alpha},$$

$$C_\alpha = \left(2 \int_0^\infty x^{-\alpha} \sin x dx \right)^{-1} = \frac{1}{\pi} \Gamma(\alpha) \sin \frac{\pi\alpha}{2}.$$

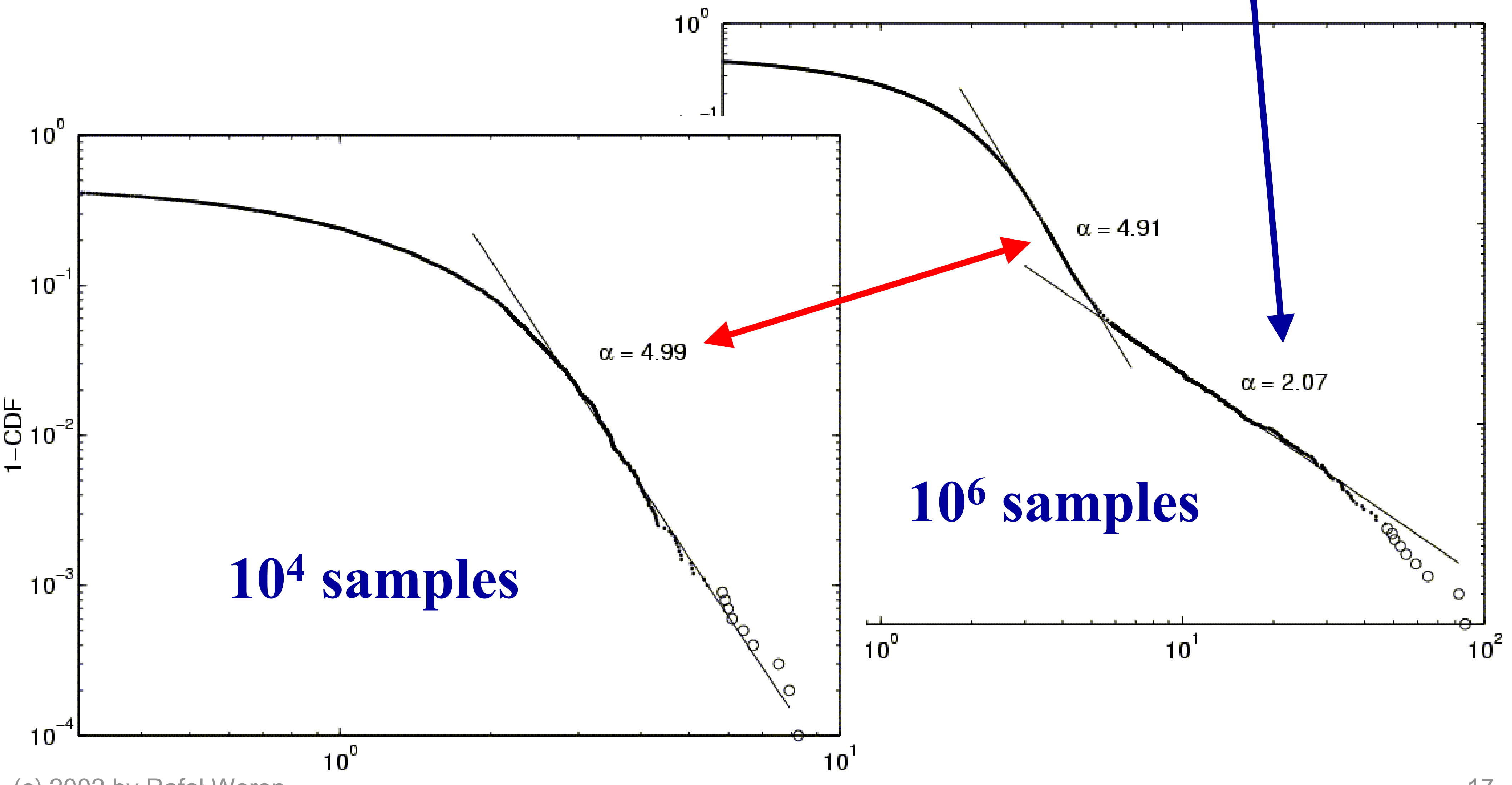
Power-law tails:
Rate of convergence



Power-law tails: Empirical analysis for α=1.8

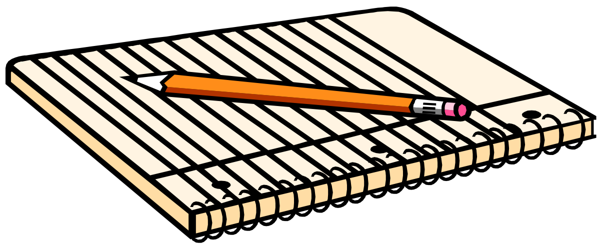


Power-law tails: Empirical analysis for α=1.95



Topics

- Introduction
- Properties of stable laws
- **Computer simulation of stable variables**
  - **Zolotarev's integral representations**
  - **The Chambers-Mallows-Stuck method**
- Estimation of parameters
- Other interesting topics



Zolotarev's integral representations

- Zolotarev's B representation $S^2_\alpha(\sigma_2,\beta_2,\mu)$

$$\log\phi(t) = \begin{cases} -\sigma_2^\alpha |t|^\alpha \exp\{-i\beta_2 sign(t)\frac{\pi}{2}K(\alpha)\} + i\mu t, & \alpha \neq 1, \\ -\sigma_2|t|\{\frac{\pi}{2} + i\beta_2 sign(t)\log|t|\} + i\mu t, & \alpha = 1, \end{cases}$$

  - with $K(\alpha) = \alpha - 1 + \text{sign}(1-\alpha)$
  - Relation between $S_\alpha(\sigma,\beta,\mu)$ and $S^2_\alpha(\sigma_2,\beta_2,\mu)$ for $\alpha\neq 1$:

$$\tan\left(\beta_2 \frac{\pi K(\alpha)}{2}\right) = \beta \tan\frac{\pi\alpha}{2}, \quad \sigma_2 = \sigma\left(1 + \beta^2\tan^2\frac{\pi\alpha}{2}\right)^{1/(2\alpha)}$$

  - For $\alpha=1$: $\beta_2 = \beta$ and $\sigma_2 = \frac{2}{\pi}\sigma$

Zolotarev's integral representations cont.

- (*Zolotarev 1986, Remark 1, p. 78*). CDF $F(x,\alpha,\beta_2)$ of S^2_α

if $\alpha \neq 1$ *and* $x > 0$ *then*

$$F(x,\alpha,\beta_2) = C(\alpha,\beta_2) + \frac{\epsilon(\alpha)}{\pi} \int_{\gamma_0}^{\frac{\pi}{2}} \exp[-x^{\alpha/(\alpha-1)} U_\alpha(\gamma,\gamma_0)]d\gamma,$$

if $\alpha = 1$ *and* $\beta_2 > 0$ *then*

$$F(x,1,\beta_2) = \frac{1}{\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \exp[-e^{-x/\beta_2} U_1(\gamma,\beta_2)]d\gamma.$$

Zolotarev's integral representations cont.

- where

$$\epsilon(\alpha) = sign(1-\alpha), \qquad \gamma_0 = -\frac{\pi}{2}\beta_2\frac{K(\alpha)}{\alpha},$$

$$C(\alpha,\beta_2) = 1-\frac{1}{4}(1+\beta_2 K(\alpha)/\alpha)(1+\epsilon(\alpha)),$$

$$U_\alpha(\gamma,\gamma_0) = \left(\frac{\sin\alpha(\gamma-\gamma_0)}{\cos\gamma}\right)^{\alpha/(1-\alpha)} \frac{\cos(\gamma-\alpha(\gamma-\gamma_0))}{\cos\gamma},$$

$$U_1(\gamma,\beta_2) = \frac{\frac{\pi}{2}+\beta_2\gamma}{\cos\gamma}\exp\left(\frac{1}{\beta_2}\left(\frac{\pi}{2}+\beta_2\gamma\right)\tan\gamma\right).$$

Chambers-Mallows-Stuck method

- Zolotarev (1956-1966), Chernin-Ibragimov (1959)
- Kanter (1975) – method for simulating $S_{\alpha<1}(1,1,0)$
- Chambers-Mallows-Stuck (1976)
  - Symmetric α-stable (SαS) case (for α=2 reduces to the Box-Muller method for Gaussian r.v.):

- generate a random variable V uniformly distributed on $(-\frac{\pi}{2}, \frac{\pi}{2})$ and an independent exponential random variable W with mean 1;

- compute

$$X = \frac{\sin(\alpha V)}{(\cos(V))^{1/\alpha}} \times \left(\frac{\cos(V - \alpha V)}{W}\right)^{(1-\alpha)/\alpha}. \tag{3.8}$$

Chambers-Mallows-Stuck method

explicit proof in Weron(1996)

- generate a random variable V uniformly distributed on $(-\frac{\pi}{2}, \frac{\pi}{2})$ and an independent exponential random variable W with mean 1;

- for $\alpha \neq 1$ compute:

$$X = S_{\alpha,\beta} \times \frac{\sin(\alpha(V + B_{\alpha,\beta}))}{(\cos(V))^{1/\alpha}} \times \left(\frac{\cos(V - \alpha(V + B_{\alpha,\beta}))}{W}\right)^{(1-\alpha)/\alpha}, \qquad (3)$$

where

$$B_{\alpha,\beta} = \frac{\arctan(\beta \tan \frac{\pi\alpha}{2})}{\alpha},$$
$$S_{\alpha,\beta} = \left[1 + \beta^2 \tan^2 \frac{\pi\alpha}{2}\right]^{1/(2\alpha)};$$

- for $\alpha = 1$ compute:

$$X = \frac{2}{\pi}\left[\left(\frac{\pi}{2} + \beta V\right) \tan V - \beta \log\left(\frac{\frac{\pi}{2} W \cos V}{\frac{\pi}{2} + \beta V}\right)\right]. \qquad (4)$$

Chambers-Mallows-Stuck method: comments

- If $X \sim S_\alpha(1,\beta,0)$ then $Y \sim S_\alpha(\sigma,\beta,\mu)$, where

$$Y = \begin{cases} \sigma X + \mu, & \alpha \neq 1, \\ \sigma X + \frac{2}{\pi}\beta\sigma\log\sigma + \mu, & \alpha = 1, \end{cases}$$

- Other approaches
  - Janicki-Kokoszka (1992) – using LePage (1980, 1987) series expansion
  - Mantegna (1994) – using Bergstrom (1952) series expansion

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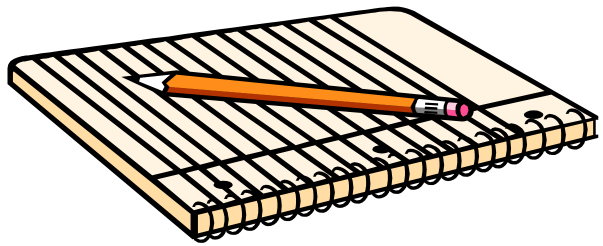
- **Estimation of parameters**
  - **Estimation of the tail index (α)**
  - **Quantile methods**
  - **Characteristic function based methods**
  - **Maximum Likelihood Method**
- Other interesting topics



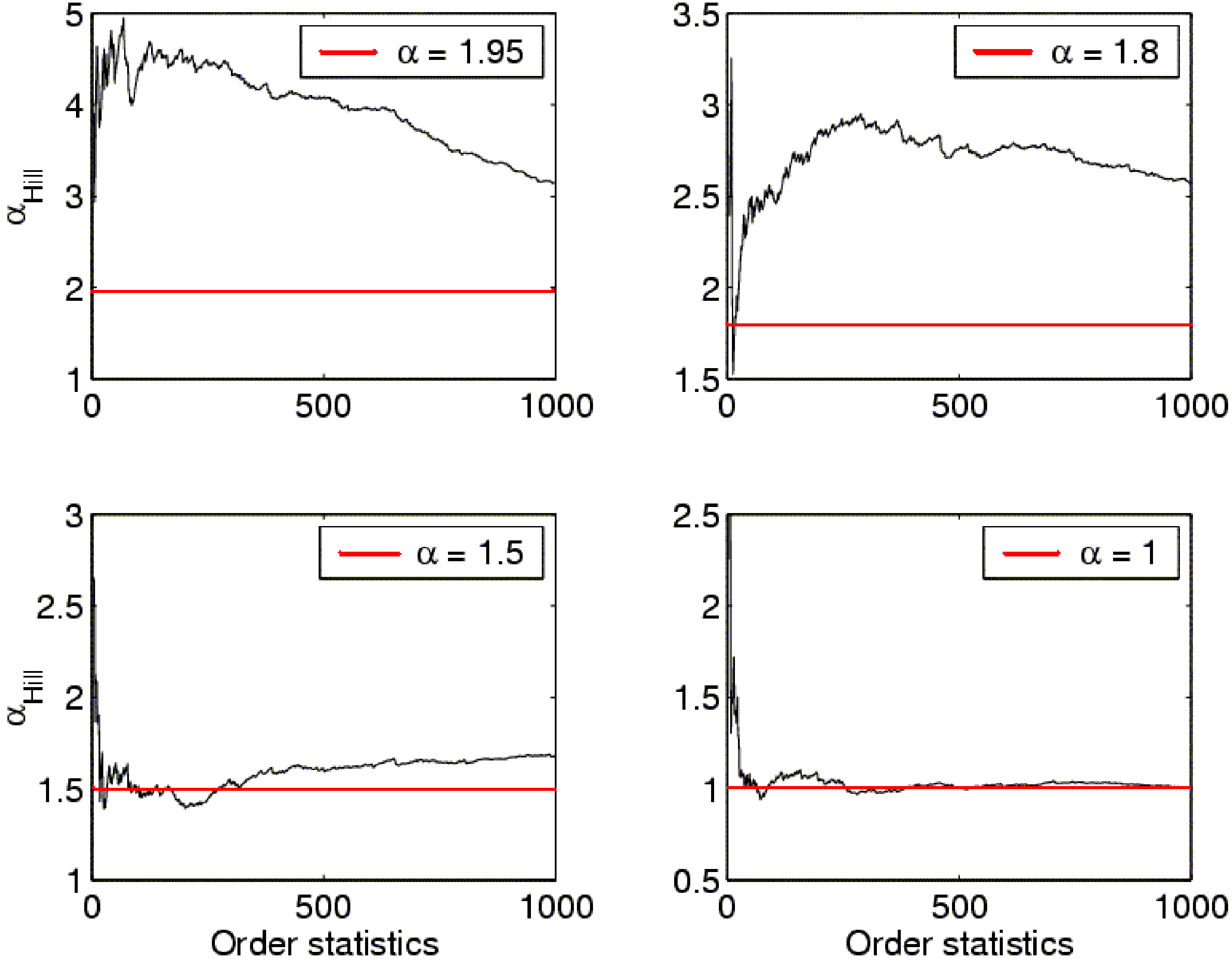
Estimation of the tail index α

- Hill (1975)

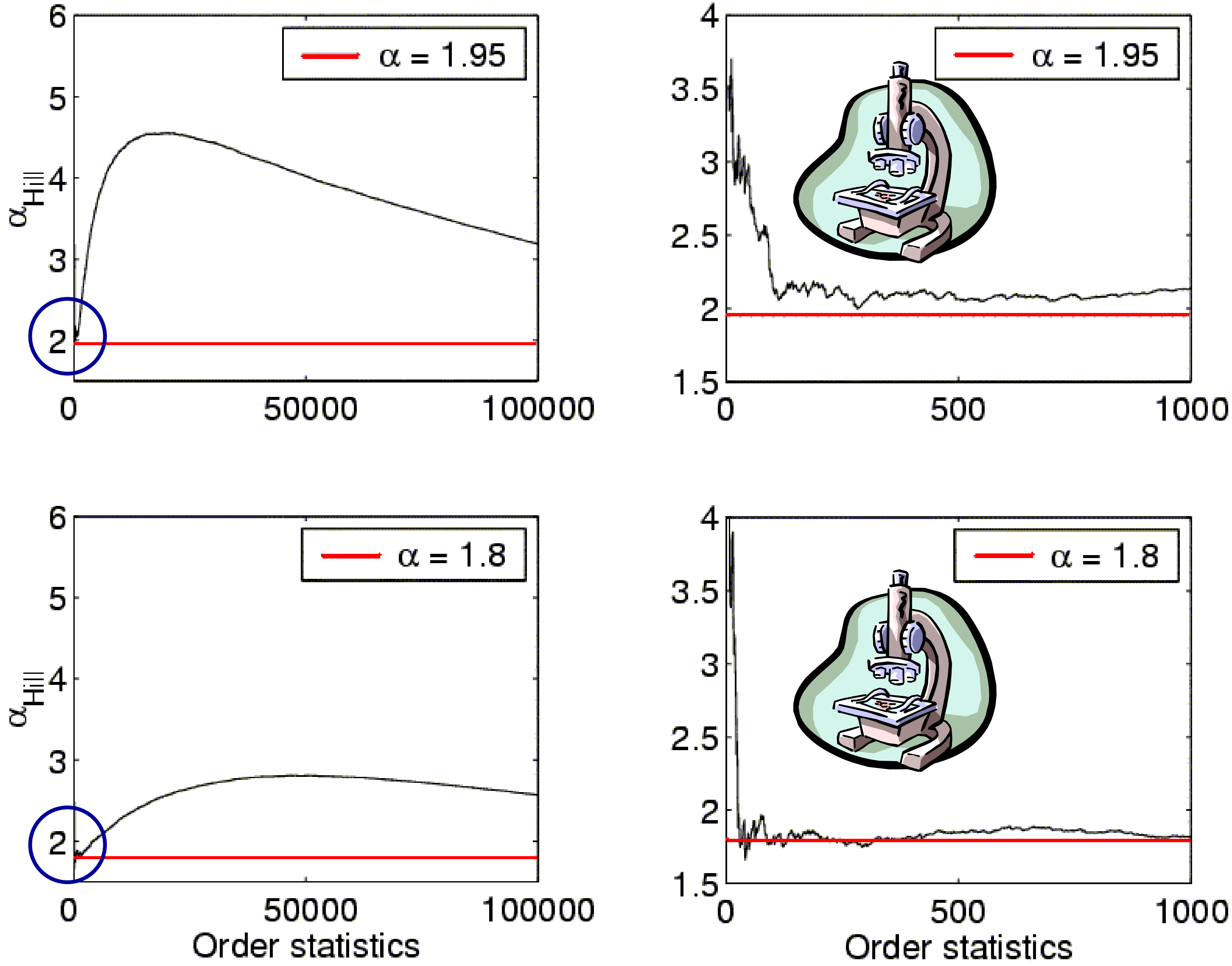
$$\alpha_{Hill}(k) = \left(\frac{1}{k} \sum_{n=1}^{k} \log \frac{X_{(n)}}{X_{(k+1)}} \right)^{-1}$$

  where $X_{(n)}$ is the n-th order statistics
- Other methods
  - Pickand (1975)
  - Dekkers-Einmahl-DeHaan (1989)
- Optimal choice of k (*usually in the vicinity of the plateau*)
  - Beirlant-Vynckier-Teugels (1996)
  - Drees-Kaufman (1998)

Performance of the Hill estimator – 10^4 samples



Performance of the Hill estimator – 10^6 samples



Sample quantile methods

Let x_f be the f–th population *quantile*, so that $S_\alpha(\sigma, \beta, \mu)(x_f) = f$. Let $\hat{x}_f$ be the corresponding *sample quantile*, i.e. $\hat{x}_f$ satisfies $F_n(\hat{x}_f) = f$.

- Fama-Roll (1968, 1971) $\hat{\sigma} = \dfrac{\hat{x}_{0.72} - \hat{x}_{0.28}}{1.654}$

$$S_{\hat{\alpha}}\left(\frac{\hat{x}_f - \hat{x}_{1-f}}{2\hat{\sigma}}\right) = f \;=\; 0.95, 0.96, 0.97$$

- **McCulloch (1986)** – estimators of all parameters for $0.6<\alpha\leq 2$

$$v_\alpha = \frac{x_{0.95} - x_{0.05}}{x_{0.75} - x_{0.25}} \qquad v_\beta = \frac{x_{0.95} + x_{0.05} - 2x_{0.50}}{x_{0.95} - x_{0.05}}$$

$$\alpha = \psi_1(v_\alpha, v_\beta)$$

v_α	v_β						
	0.0	0.1	0.2	0.3	0.5	0.7	1.0
2.439	2.000	2.000	2.000	2.000	2.000	2.000	2.000
2.5	1.916	1.924	1.924	1.924	1.924	1.924	1.924
2.6	1.808	1.813	1.829	1.829	1.829	1.829	1.829
2.7	1.729	1.730	1.737	1.745	1.745	1.745	1.745
2.8	1.664	1.663	1.663	1.668	1.676	1.676	1.676
3.0	1.563	1.560	1.553	1.548	1.547	1.547	1.547
3.2	1.484	1.480	1.471	1.460	1.448	1.438	1.438
3.5	1.391	1.386	1.378	1.364	1.337	1.318	1.318
[illegible]	[illegible]	1.273	1.266	1.250	1.210	1.184	1.150
[illegible]	[illegible]	1.121	1.114	1.101	1.067	1.027	0.973
[illegible]	1.029	1.021	1.014	1.004	0.974	0.935	0.874
[illegible]	0.896	0.892	0.887	0.883	0.855	0.823	0.769
10.0	0.818	0.812	0.806	0.801	0.780	0.756	0.691
15.0	0.698	0.695	0.692	0.689	0.676	0.656	0.595
25.0	0.593	0.590	0.588	0.586	0.579	0.563	0.513

Characteristic function based methods

- For a sample $x_1,\ldots,x_n$ define sample cf: $\hat{\phi}(t) = \frac{1}{n}\sum_{j=1}^{n} e^{itx_j}$
- Press (1972) – method of moments
  - for a choice of $t_1,\ldots,t_4$

$$\hat{\alpha} = \frac{\log \frac{\log|\hat{\phi}(t_1)|}{\log|\hat{\phi}(t_2)|}}{\log\left|\frac{t_1}{t_2}\right|} \qquad \hat{\beta} = \frac{\frac{\hat{u}(t_4)}{t_4} - \frac{\hat{u}(t_3)}{t_3}}{[|t_4|^{\hat{\alpha}-1} - |t_3|^{\hat{\alpha}-1}]\hat{\sigma}^{\alpha}\tan\frac{\hat{\alpha}\pi}{2}},$$

$$\log\hat{\sigma} = \frac{\log|t_1|\log(-\log|\hat{\phi}(t_2)|) - \log|t_2|\log(-\log|\hat{\phi}(t_1)|)}{\log\left|\frac{t_1}{t_2}\right|} \qquad \hat{\mu} = \frac{|t_4|^{\hat{\alpha}-1}\frac{\hat{u}(t_3)}{t_3} - |t_3|^{\hat{\alpha}-1}\frac{\hat{u}(t_4)}{t_4}}{|t_4|^{\hat{\alpha}-1} - |t_3|^{\hat{\alpha}-1}}.$$

- Press (1972), Leitch-Paulson (1975) – minimum distance method

$$h(\alpha,\sigma,\beta,\mu) = \int_{-\infty}^{\infty} |\phi(t) - \hat{\phi}(t)|^r W(t)\,dt.$$

Characteristic function based methods: regression

- **Koutrouvelis (1980, 1981)**
- From the definition of the cf in representation $S_\alpha(\sigma,\beta,\mu)$:

$$\log(-\log|\phi(t)|^2) = \log(2\sigma^\alpha) + \alpha\log|t|$$

$$\arctan\left(\frac{Im\phi(t)}{Re\phi(t)}\right) = \mu t + \beta\sigma^\alpha \tan\frac{\pi\alpha}{2} sign(t)|t|^\alpha$$

- Thus to estimate α and σ we can:

$$\text{regress} \quad y = \log(-\log|\phi_n(t)|^2) \text{ on } w = \log|t|$$

$$y_k = m + \alpha w_k + \epsilon_k, \quad k = 1, 2, \ldots, K,$$

- **Kogon-Williams (1998)** used the $S^0_\alpha(\sigma,\beta,\mu_0)$ representation and performed an initial scale-location normalization

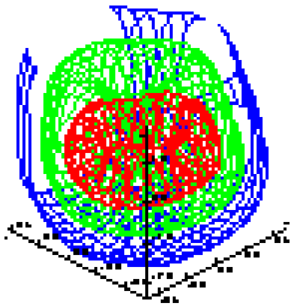
Maximum Likelihood Method

- DuMouchel (1971-83), Nolan (1997-2002)
  - SαS case: Brorsen-Yang (1990), McCulloch (1998)

$$f_\alpha(x) = \frac{\alpha}{\pi|1-\alpha|} x^{1/(\alpha-1)} \int_0^{\pi/2} U_\alpha(\gamma,0) e^{-x^{\alpha/(\alpha-1)} U_\alpha(\gamma,0)} d\gamma$$

- STABLE (www.academic2.american.edu/~jpnolan/stable/stable.html) uses adaptive quadrature DQDAG to evaluate the integrals in the general asymmetric case
- XploRe 4.6 (www.xplore-stat.de) includes estimation methods
  - Net based data analysis
  - Japanese version (www.d-tec.co.jp/products/xplore/index.html)

MD*TECH Method & Data Technologies



Choosing the best method

Fastest

Quantile method (McCulloch)

Least accurate

CF regression (Koutrouvelis-Kogon-Williams)

Slowest

Most accurate

MLE (Nolan)

Topics

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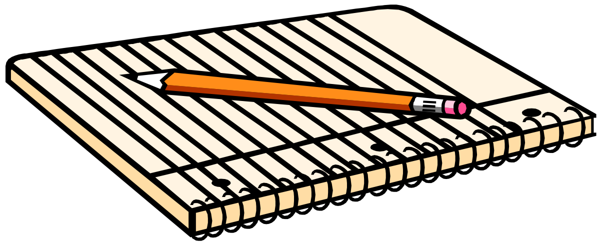
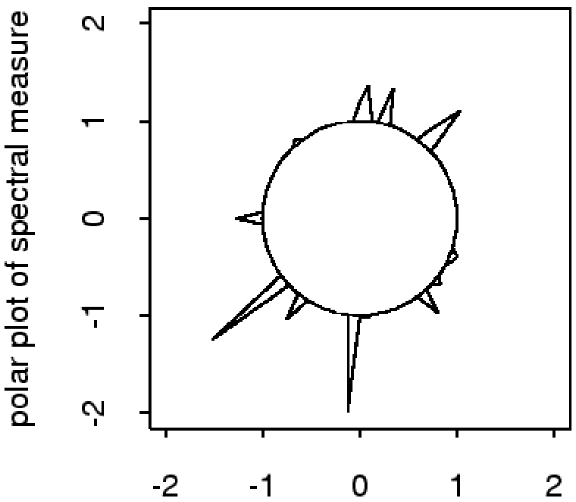
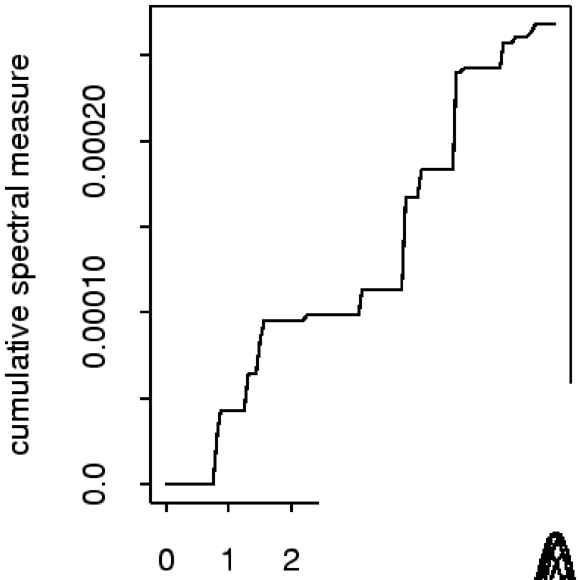
  - **Multivariate α-stable laws**
  - **α-stable stochastic processes**

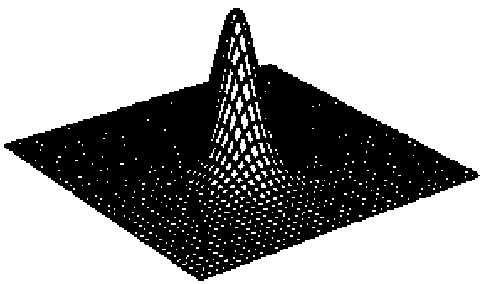


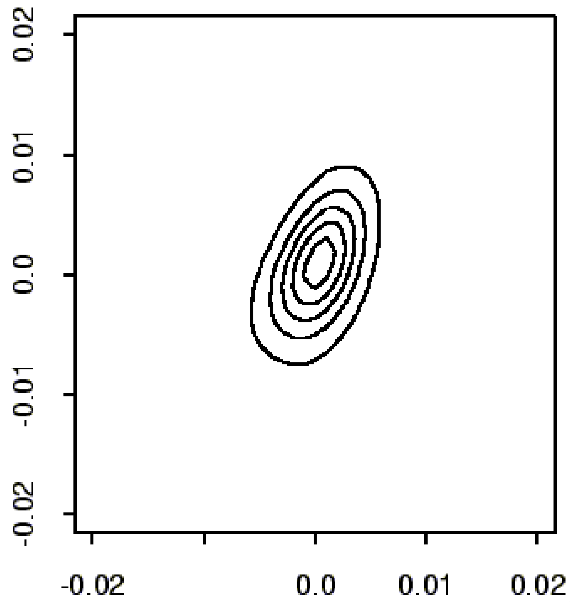
Multivariate stable laws

- Bivariate stable fit to DEM/GBP and JPY/GBP (Nolan, 1999)









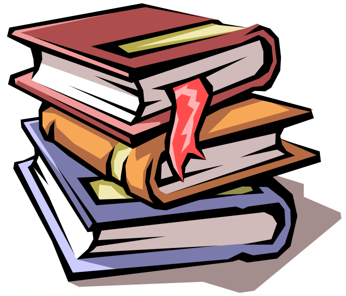
  - McCulloch (1999)
  - Davydov (2001)

α-stable stochastic processes

- Ito-McKean (1965), Lukacs (1967), Breiman (1968) – *α-stable motion (i.e. process with α-stable, independent and stationary increments – called an α-stable Levy process in math literature)*
- Rootzen (1978), Kokoszka-Taqqu (1994) – *α-stable (F)AR(I)MA*
- Hardin (1982), Samorodnitsky-Taqqu (1990) – *spectral represent.*
- Cambanis-Soltani (1984) – *prediction of harmonizable processes*
- Cambanis-Hardin-Weron (1987), Takashima (1989) – *ergodicity*
- Rosinski (1995) – *structure of stationary SαS processes*
- Kono-Maejima (1991), Takenaka (1991), Burnecki-Rosinski-Weron (1997), Maejima-Sato (1999) – *self-similar processes*
- Mantegna-Stanley (1994), Koponen (1995) – *TLF's*

Stable distributions and processes provide models for heavy-tailed data

References



- See:

  - R.Weron (1996) "On the Chambers-Mallows-Stuck Method for Simulating Skewed Stable Random Variables", Statistics & Probability Letters 28, 165-171. *See also:* R.Weron (1996) "Correction to: On the Chambers-Mallows-Stuck Method for Simulating Skewed Stable Random Variables", Research Report HSC/96/1, Wroclaw University of Technology.
  - R.Weron (2001) "Levy-stable distributions revisited: tail index > 2 does not exclude the Levy-stable regime", International Journal of Modern Physics C 12 (2), 209-223, [cond-mat/0103256].
  - … and references therein.

- These and related papers can be downloaded from:
  - www.im.pwr.wroc.pl/~hugo/Publications.html