Network Data Envelopment Analysis (NDEA): Bottom-up versus top-down approach

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Overview

- From DEA to Network DEA (NDEA)
- A taxonomy of NDEA approaches for series processes
- Methods with pros and cons
- Joint graphical and MOP representation
- Summary
- References

DEA in general

DEA is a non-parametric technique for evaluating the relative efficiency of a set of peer entities, called Decision Making Units (DMUs), which use multiple incommensurable inputs to produce multiple incommensurable outputs.

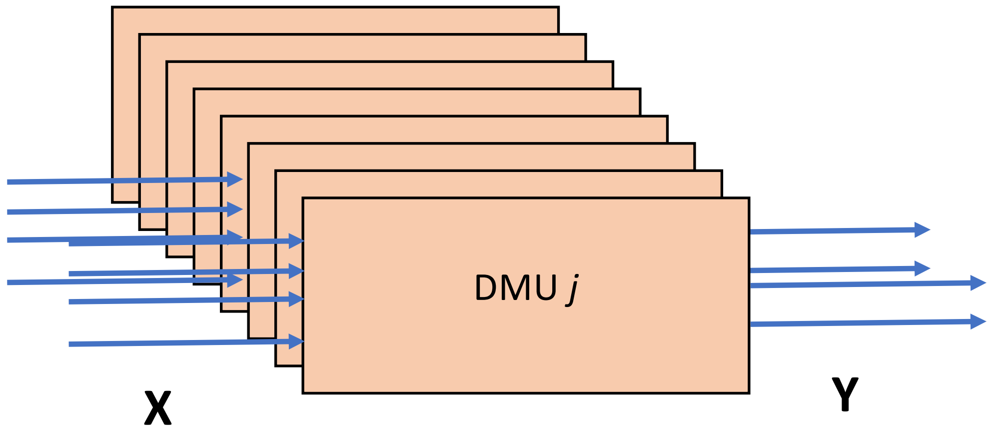
No assumption about the production function

The DMUs are assumed homogeneous

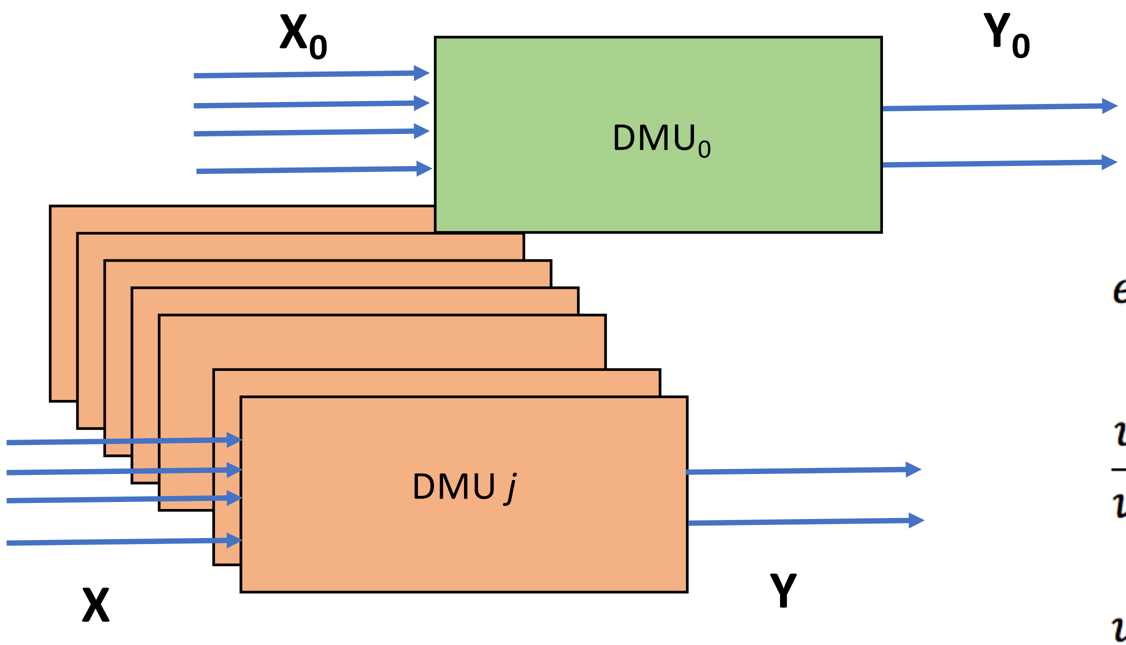
Linear Programming (LP) is the mathematical instrument used

Standard DEA

The DMUs are considered as “black boxes”



Standard DEA (multiplier representation)



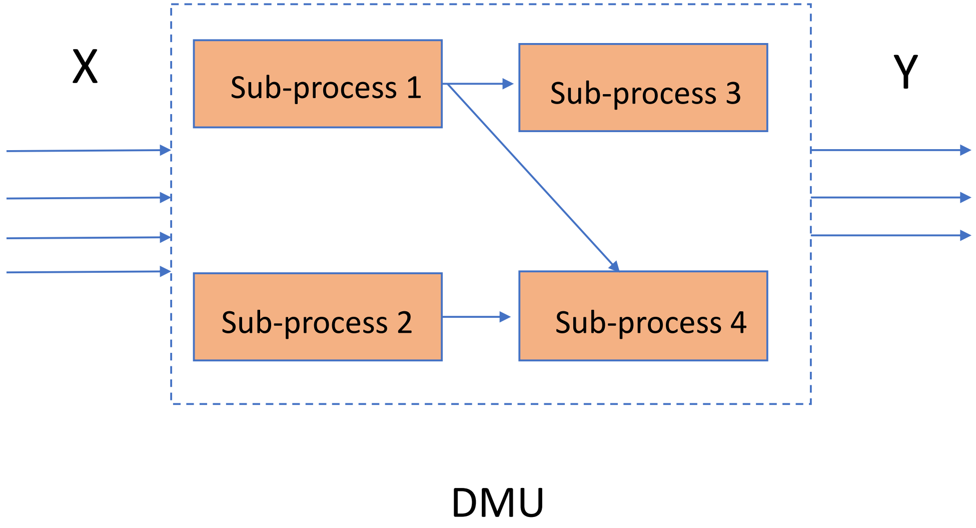
$$e_0 = max \frac{uY_0}{vX_0}$$

$$\frac{uY_j}{vX_j} \leq 1, \qquad j = 1, \ldots, n$$

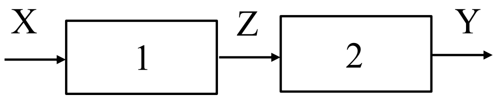
$$u, v \geq 0$$

Network DEA

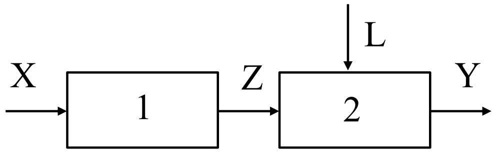
The DMU is viewed as a network of sub-process (divisions, stages etc.) that are connected via the flow of intermediate measures



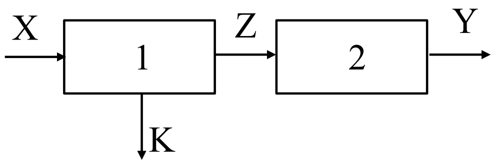
The 4 types of two-stage processes



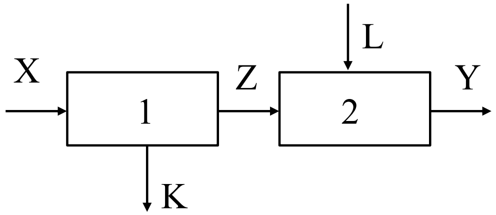
Type I



Type II

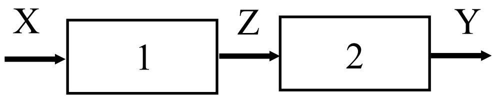


Type III



Type IV

The simple two-stage process

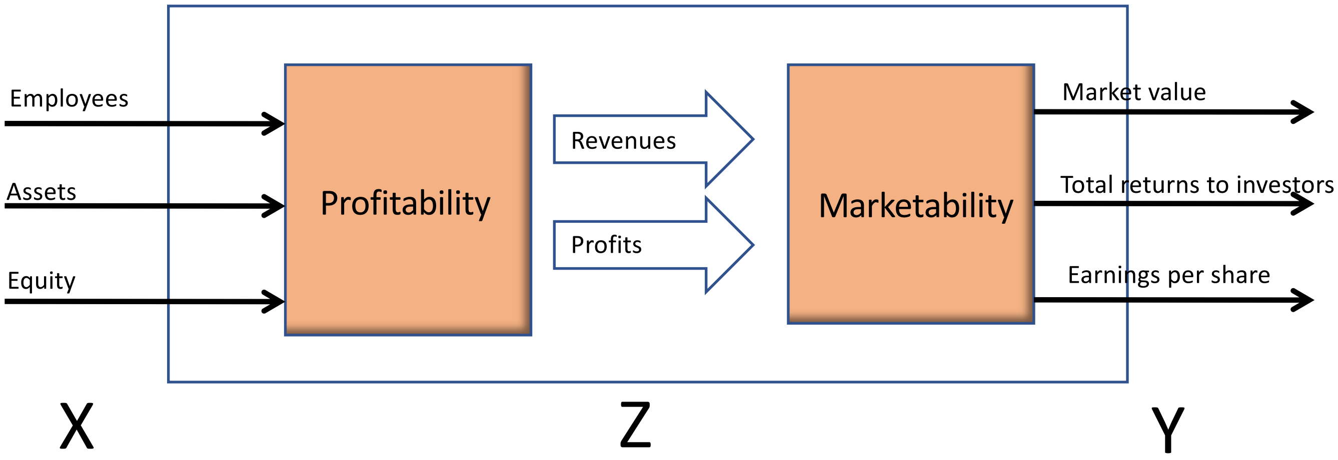


Stage 1 Stage 2

$$e_j^1 = \frac{\varphi Z_j}{\eta X_j} \ , e_j^2 = \frac{\omega Y_j}{\varphi Z_j}$$

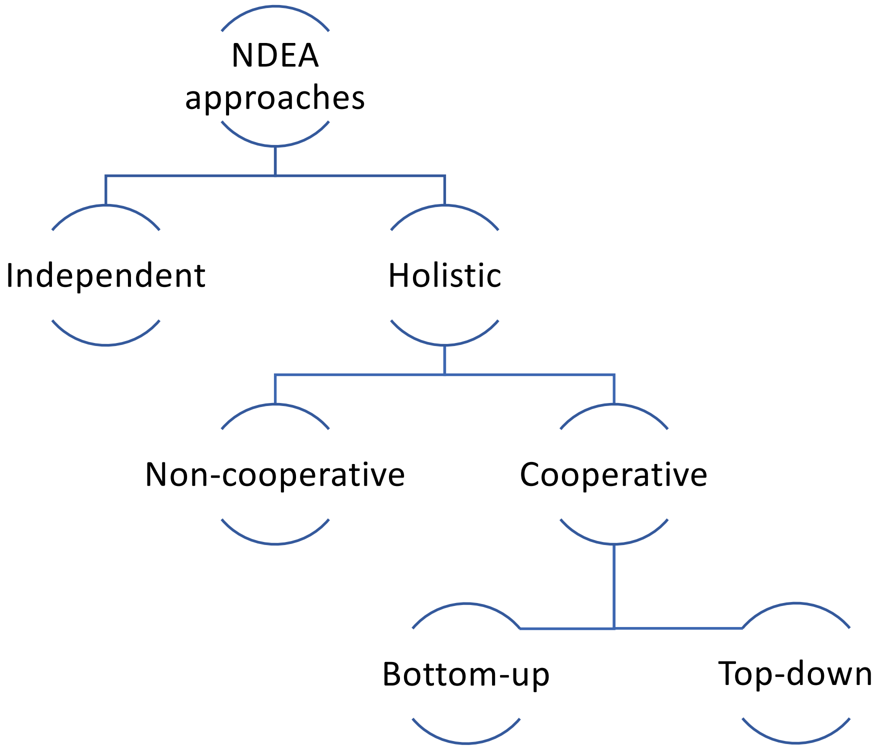
The definition of the system overall efficiency is **not universal**

Measuring profitability and marketability of the top 55 U.S. commercial banks (Seiford and Zhu, 1999)

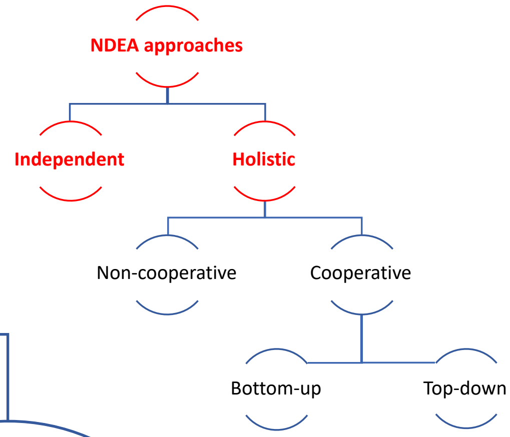


The bank production process

A taxonomy of NDEA approaches for series processes



Independent Vs. Holistic approach

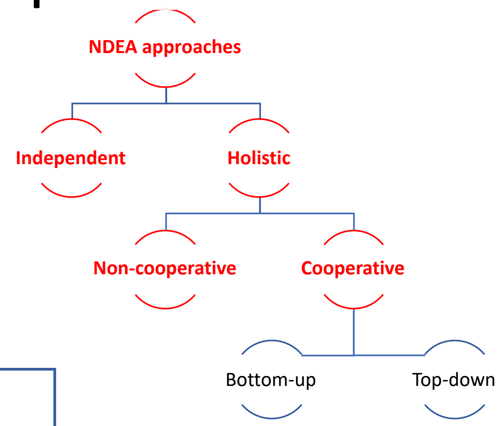


NDEA approaches

Independent: The efficiency of the individual stages and the overall efficiency of the system are assessed independently. Standard DEA is applied.

Holistic: The efficiencies of the individual stages and the overall system efficiency are estimated jointly by considering the interdependencies of the stages

Non-cooperative Vs. cooperative approach

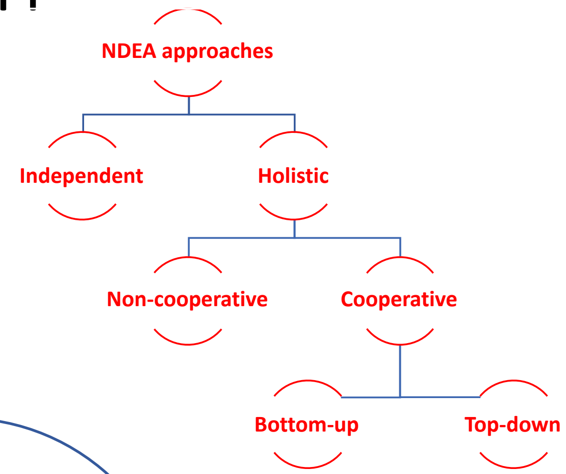


Holistic

Non-cooperative: Pre-emptive priority is given to one of the two stages (leader stage), whose efficiency is assessed first. Then, the efficiency of the other stage (follower) is assessed in a manner that the optimal efficiency of the leader is maintained

Cooperative: The stage and the system efficiencies are estimated simultaneously.

Top-down Vs. bottom-up approach



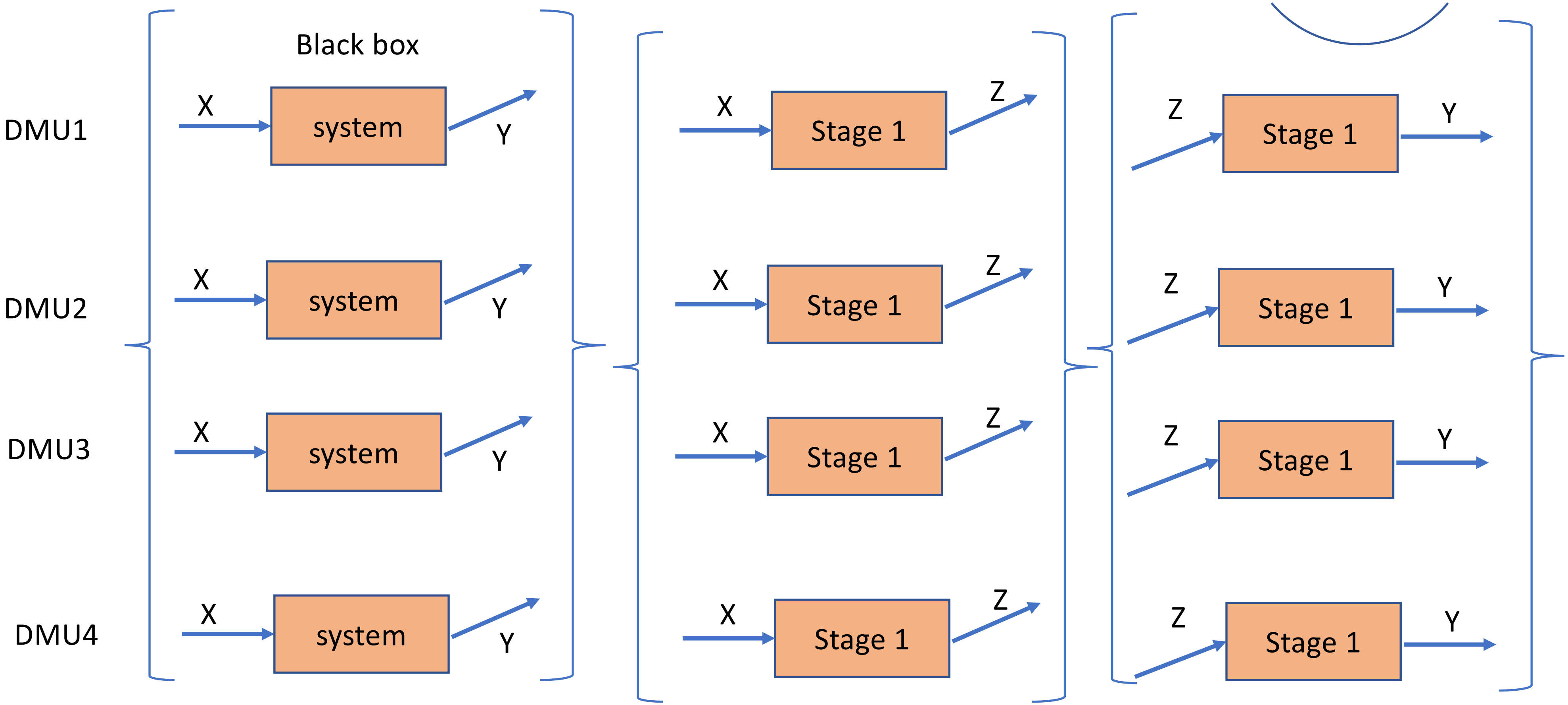
Cooperative

Bottom-up: The stages are given priority for optimization

Top-down: The system is given priority for optimization

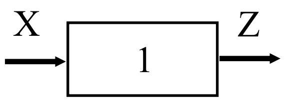
Independent approach

Independent: The efficiency of the individual stages and the overall efficiency of the system are assessed independently. Standard DEA is applied.



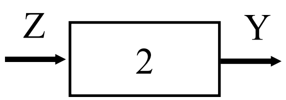
Independent assessment

Stage 1



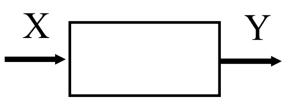
$$E_{j_0}^1 = max\frac{\varphi Z_{j_0}}{\eta X_{j_0}}$$
$$s.t.$$
$$\varphi Z_j - \eta X_j \le 0, \quad j = 1, \dots, n$$
$$\eta \ge \varepsilon, \varphi \ge \varepsilon$$

Stage 2



$$E_{j_0}^2 = max\frac{\omega Y_{j_0}}{\varphi Z_{j_0}}$$
$$s.t.$$
$$\omega Y_j - \varphi Z_j \le 0, \quad j = 1, \dots, n$$
$$\varphi \ge \varepsilon, \ \omega \ge \varepsilon$$

System



$$E_{j_0}^0 = max\frac{\omega Y_{j_0}}{\eta X_{j_0}}$$
$$s.t.$$
$$\omega Y_j - \eta X_j \le 0, \quad j = 1, \dots, n$$
$$\omega \ge \varepsilon, \eta \ge \varepsilon$$

Standard DEA is applied to each stage and the system as a block-box separately, ignoring their connection via the intermediate measures Z

Pros
Easy to implement

Cons
Controversial results:
Efficient system with inefficient divisions
Inefficient system with efficient divisions

Inter-DMU dominance property violated:
Higher system efficiency with lower divisional efficiencies

Holistic approach

The efficiencies of the individual stages and the overall system efficiency are estimated jointly by considering the interdependencies of the stages

The weights associated with the intermediate measures are assumed to be the same regardless their role in the system

Non-cooperative
The leader-follower paradigm (Liang, Cook & Zhu, 2008)

Non-cooperative: Pre-emptive priority is given to one of the two stages (leader stage), whose efficiency is assessed first. Then, the efficiency of the other stage (follower) is assessed in a manner that the optimal efficiency of the leader is maintained

$$e_{j_o}^{1*} = max \frac{\varphi Z_{j_0}}{\eta X_{j_0}}$$
$$s.t.$$
$$\varphi Z_j - \eta X_j \leq 0, \quad j = 1, \dots, n$$
$$\eta \geq \varepsilon, \varphi \geq \varepsilon$$

$$e_{j_o}^{2*} = max \frac{\omega Y_{j_0}}{\varphi Z_{j_0}}$$
$$s.t.$$
$$\varphi Z_{j_0} - e_{j_o}^{1*} \times \eta X_{j_0} \geq 0$$
$$\omega Y_j - \varphi Z_j \leq 0, \quad j = 1, \dots, n$$
$$\eta \geq \varepsilon, \varphi \geq \varepsilon$$

$$e_{j_o}^{o} = e_{j_o}^{1*} \times e_{j_o}^{2*}$$

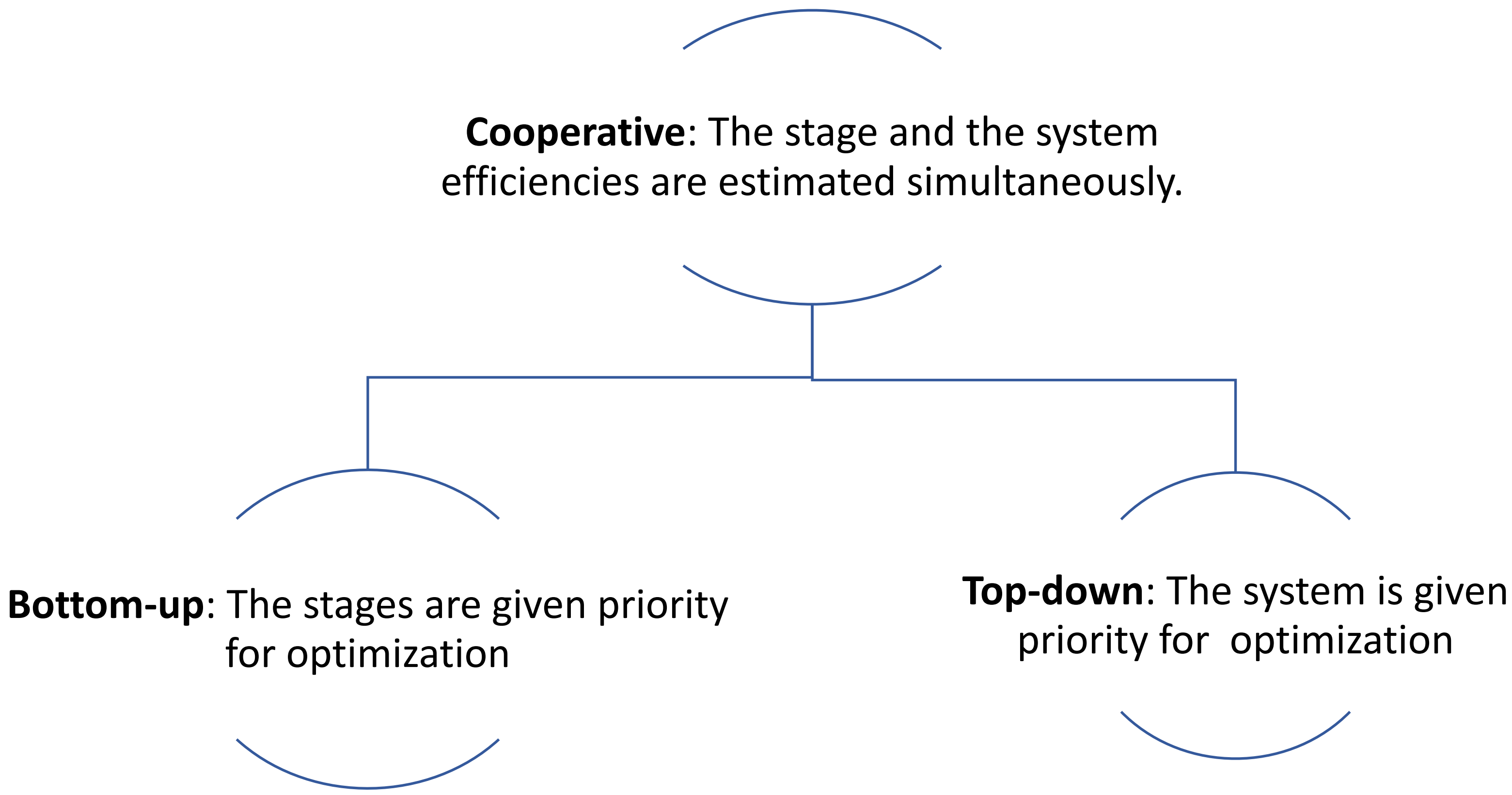
It is a lexicographic programing approach

External information is needed to define the leader

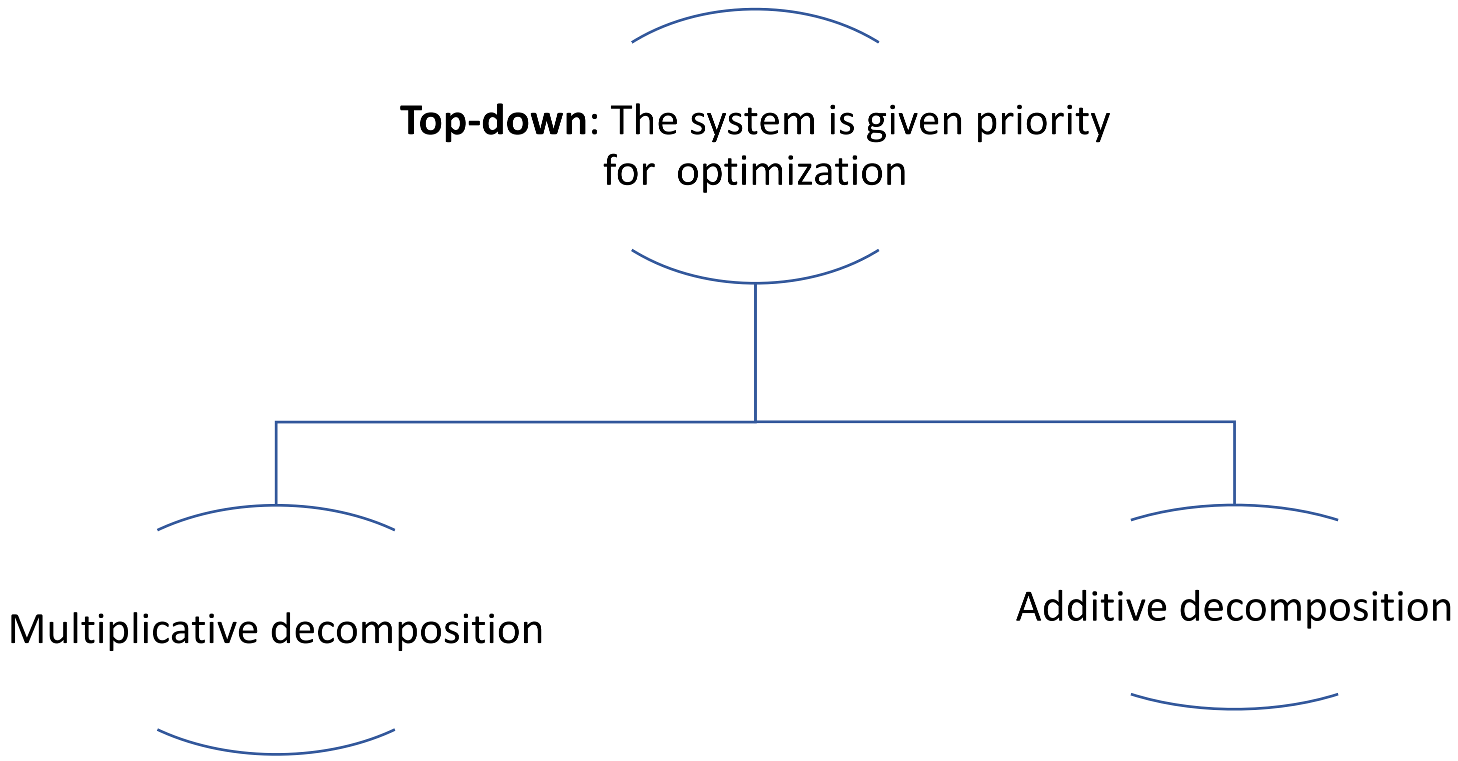
The overall system efficiency is obtained ex post as the product of the stage efficiencies and depends on the selection of the leader.

Locates an extreme point on the Pareto front in the divisional efficiencies space (e1,e2). Compliance with intra and inter DMU dominance property.

Cooperative approach



Top-down approach



Multiplicative decomposition (Kao & Hwang, 2008)

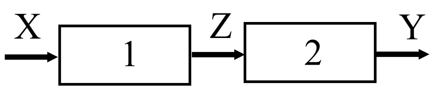
$$e_{j_0}^{o} = \max \frac{\omega Y_{j_0}}{\eta X_{j_0}}$$

$$s.t.$$

$$\frac{\varphi Z_j}{\eta X_j} \le 1, \quad j = 1, \ldots, n$$

$$\frac{\omega Y_j}{\varphi Z_j} \le 1, \quad j = 1, \ldots, n$$

$$\varphi \ge 0, \eta \ge 0, \omega \ge 0$$



The system efficiency is defined as the squared geometric mean of the divisional efficiencies.

The system efficiency is optimized whereas the divisional efficiencies are obtained as offspring.

Pros

Easy to implement

Cons

Decomposition not unique

Inter-DMU and intra-DMU dominance property violated in network structures of types II, III and IV. It is likely to get a suboptimal solution in terms of the system efficiency with higher divisional efficiencies.

Additive decomposition (Chen, Cook, Li, Zhu, 2009)

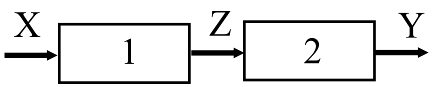
$$e_{j_0}^{o} = \max \frac{\omega Y_{j_0} + \varphi Z_{j_0}}{\eta X_{j_0} + \varphi Z_{j_0}}$$

$s.t.$

$$\frac{\varphi Z_j}{\eta X_j} \le 1,\; j = 1,\dots,n$$

$$\frac{\omega Y_j}{\varphi Z_j} \le 1,\; j = 1,\dots,n$$

$$\eta \ge 0, \varphi \ge 0, \omega \ge 0$$



The system efficiency is defined as a weighted average of the divisional efficiencies.

$$\frac{\omega Y_j + \varphi Z_j}{\eta X_j + \varphi Z_j} = t_j^1 \frac{\varphi Z_j}{\eta X_j} + t_j^2 \frac{\omega Y_j}{\varphi Z_j},\; t_j^1 + t_j^2 = 1$$

$$t_j^1 = \frac{\eta X_j}{\eta X_j + \varphi Z_j},\quad t_j^2 = \frac{\varphi Z_j}{\eta X_j + \varphi Z_j}$$

Pros

Easy to implement

Cons

Decomposition not unique

Biased assessment in favor of the second stage ($t_2 <= t_1$)

Inter-DMU and intra-DMU dominance property violated in network structures of types II, III and IV. It is likely to get a suboptimal solution in terms of the system efficiency with higher divisional efficiencies.

Bottom-up approach

Bottom-up: The stages are given priority for optimization

Min-max method

the weak-link method

Pros

Unique divisional efficiencies

Complies with Inter-DMU and intra-DMU dominance property in network structures of all types I, II, III and IV.

Cons

Computationally more demanding

The bottom-up approach is in fact a multi-objective programming approach

The difference between the two methods is in the choice of the scalarizing function employed to reach the Pareto optimal solution

The min-max method

$\min \delta$

$s.t.$

$t_1(E_{j_0}^1 - wZ_{j_0}) \leq \delta$

$t_2(E_{j_0}^2 - \frac{uY_{j_0}}{wZ_{j_0}}) \leq \delta$

$vX_{j_0} = 1$

$wZ_j - vX_j \leq 0, \ \ j = 1, \dots, n$

$uY_j - wZ_j \leq 0, \ \ j = 1, \dots, n$

$v \geq \varepsilon, w \geq \varepsilon, u \geq \varepsilon, \delta \geq 0$

The model employs the weighted Tchebycheff norm (L_∞ norm) to locate a point on the Pareto front, by minimizing the maximum of the weighted deviations

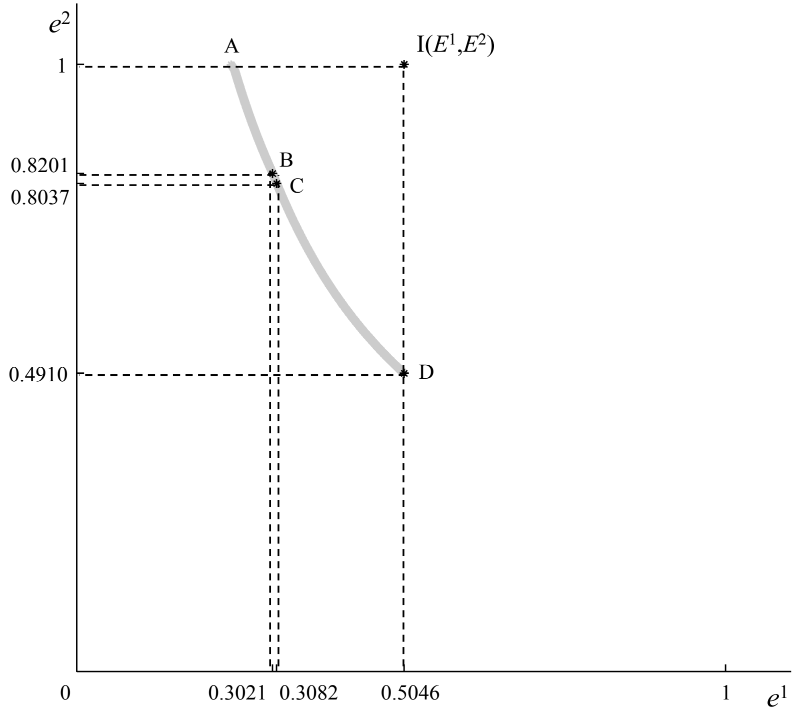
$t_1(E_{j_0}^1 - e_{j_0}^1)$ and $t_2\left(E_{j_0}^2 - e_{j_0}^2\right)$

of the divisional efficiency scores from the ideal point ($E_{j_0}^1$, $E_{j_0}^2$), with weights $t_1>0$ and $t_2>0$.

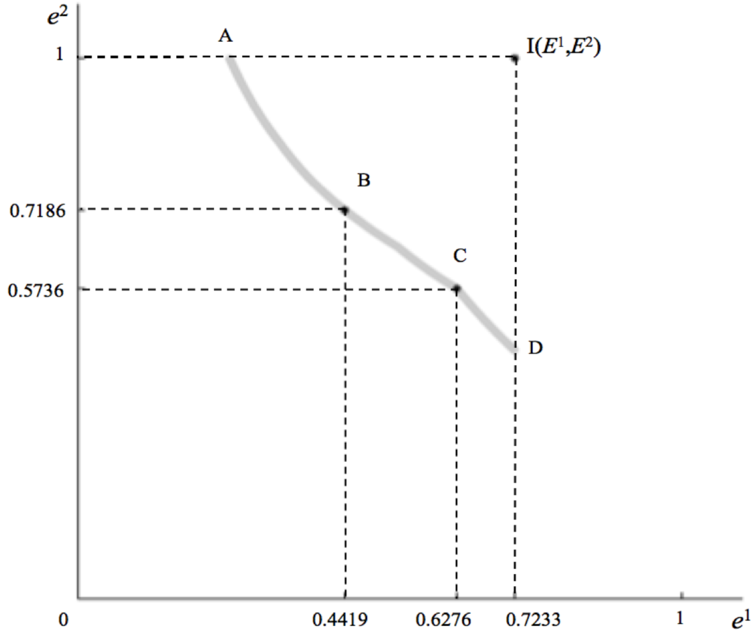
The neutral min-max model

$$\min \delta$$
$$s.t.$$
$$E^1_{j_0} - wZ_{j_0} \le \delta$$
$$\left(E^2_{j_0} - \delta\right) wZ_{j_0} - uY_{j_0} \le 0$$
$$vX_{j_0} = 1$$
$$wZ_j - vX_j \le 0, \quad j = 1, \dots, n$$
$$uY_j - wZ_j \le 0, \quad j = 1, \dots, n$$
$$v \ge \varepsilon, w \ge \varepsilon, u \ge \varepsilon, \delta \ge 0$$

The segment BD of the Pareto front consists of an infinite number of alternative efficiency decompositions (non-uniqueness) of the overall efficiency, according to the multiplicative decomposition. Contrariwise, the min-max model generates the unique pair of Pareto optimal efficiency scores depicted on point C.



The neutral min-max model



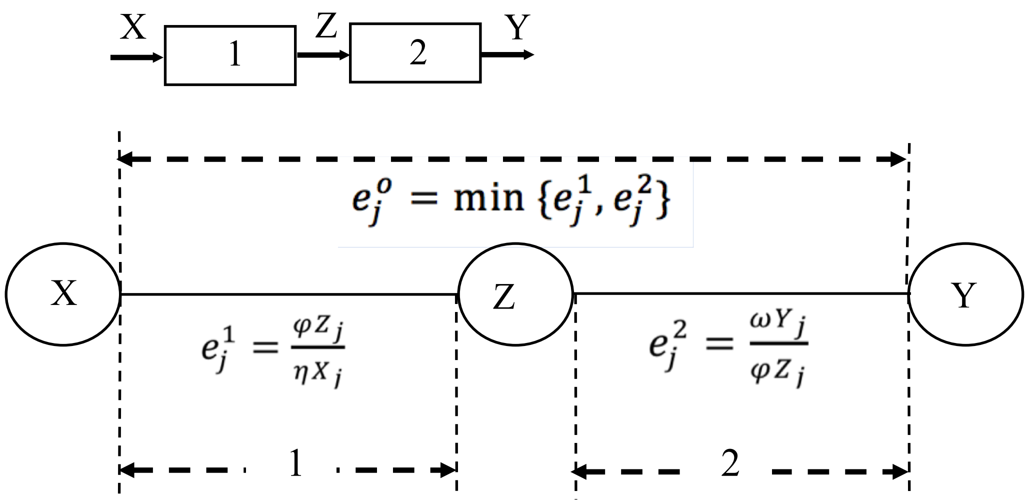
Point B is obtained by the neutral min-max model

Point C is obtained by the multiplicative decomposition method and in this case is unique

Point C is accessible by the weighted min-max model with $t_1 = 0.81668, t_2 = 0.18332$

So the multiplication method achieves the maximum system efficiency score by over-weighting the stage-1 significantly at the expense of the stage-2.

The weak-link method



Definition of overall efficiency based on a Max-flow/min-cut analogue

The problem becomes the identification of the weak link

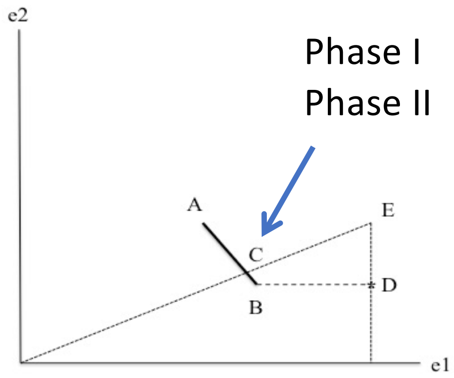
The identification of the weak link should meet two properties: a) uniqueness and b) being supported by a reasonable and meaningful search orientation

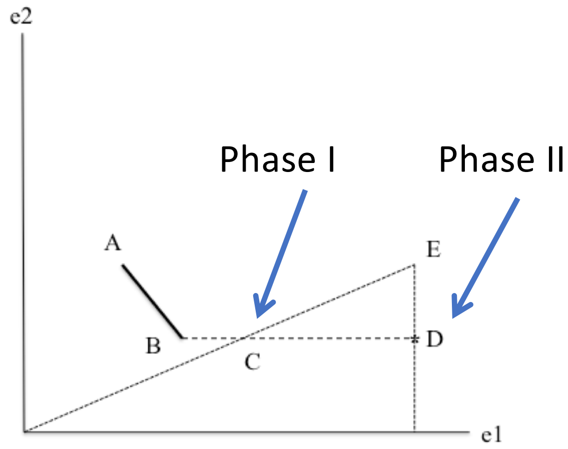
The capacities (individual efficiency scores) of the two stages are estimated in a manner that the minimal capacity (the weak link) and, thus, the overall system efficiency gets the maximum possible value.

This is accomplished with a weighted max-min formulation, which seeks to maximize the minimum weighted achievement from a zero-level efficiency:

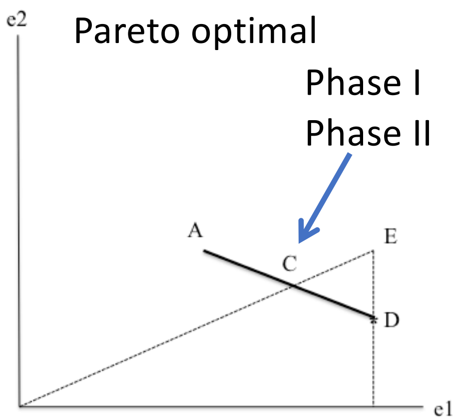
$$e_{j_o}^o = max_{v,w,u}\,[min\{q_1 e_{j_o}^1, q_2 e_{j_o}^2\}] \quad q_1 = 1/E_{j_0}^1 \qquad q_2 = 1/E_{j_0}^2$$

The two-phase solution procedure





Weak Pareto

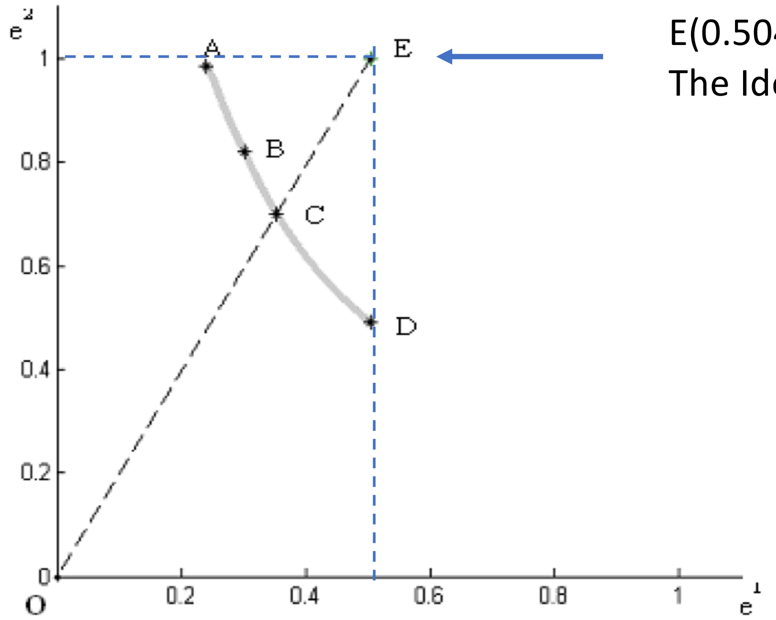


Joint graphical and MOP representation

$max\ e^o(v, w, u)$
$max\ e^1(v, w, u)$
$max\ e^2(v, w, u)$
s.t.
$e^1(v, w, u) \leq 1$
$e^2(v, w, u) \leq 1$
$v \geq 0, w \geq 0, u \geq 0$

Independent assessments

Find the ideal point (E^0, E^1,E^2) in the (e^0, e^1, e^2) space



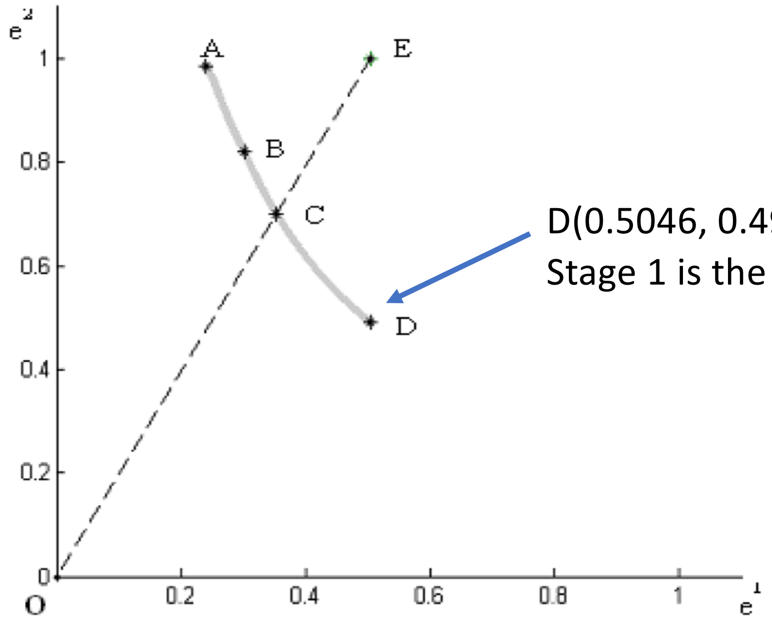
E(0.5046, 1)
The Ideal point

Joint graphical and MOP representation

$max\ e^o(v,w,u)$
$max\ e^1(v,w,u)$
$max\ e^2(v,w,u)$
s.t.
$e^1(v,w,u) \leq 1$
$e^2(v,w,u) \leq 1$
$v \geq 0, w \geq 0, u \geq 0$

Leader-follower paradigm

Lex max $\{e^1, e^2\}$ to get D



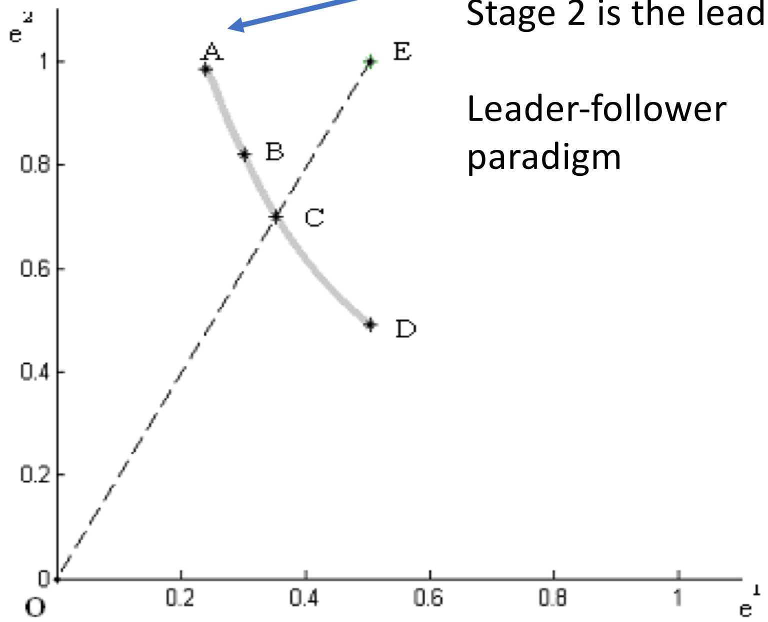
D(0.5046, 0.4910)
Stage 1 is the leader

Joint graphical and MOP representation

$$
\begin{aligned}
& max\ e^o(v,w,u) \\
& max\ e^1(v,w,u) \\
& max\ e^2(v,w,u) \\
& \text{s.t.} \\
& e^1(v,w,u) \leq 1 \\
& e^2(v,w,u) \leq 1 \\
& v \geq 0, w \geq 0, u \geq 0
\end{aligned}
$$

Leader-follower paradigm

Lex max {e^2, e^1} to get A



A(0.2378,1)
Stage 2 is the leader

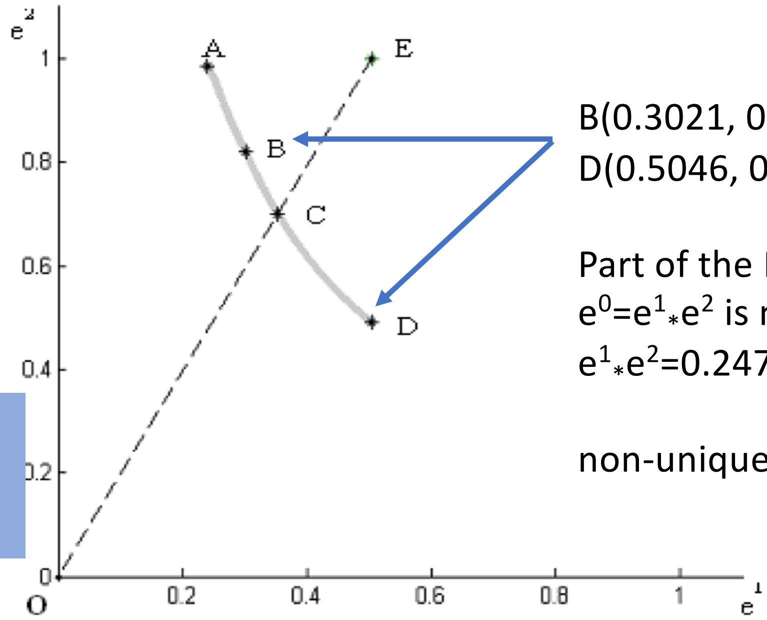
Leader-follower paradigm

Joint graphical and MOP representation

$max\ e^0(v,w,u)$
$max\ e^1(v,w,u)$
$max\ e^2(v,w,u)$
s.t.
$e^1(v,w,u) \leq 1$
$e^2(v,w,u) \leq 1$
$v \geq 0, w \geq 0, u \geq 0$

Multiplicative approach

Lex max $\{e^0, e^1\}$ to get D
and
Lex max $\{e^0, e^2\}$ to get B



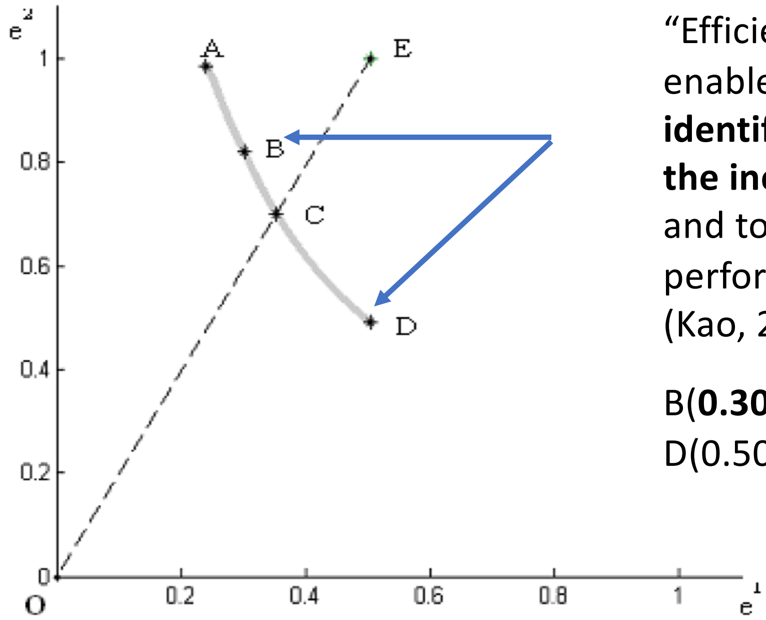
B(0.3021, 0.8201)
D(0.5046, 0.4910)

Part of the PF where
$e^0=e^1{}_*e^2$ is maximized
$e^1{}_*e^2=0.2477$

non-unique decomposition

The additive decomposition is similar but not displayed

Joint graphical representation



"Efficiency decomposition enables decision makers to **identify the stages that cause the inefficiency of the system**, and to effectively improve the performance of the system" (Kao, 2014)

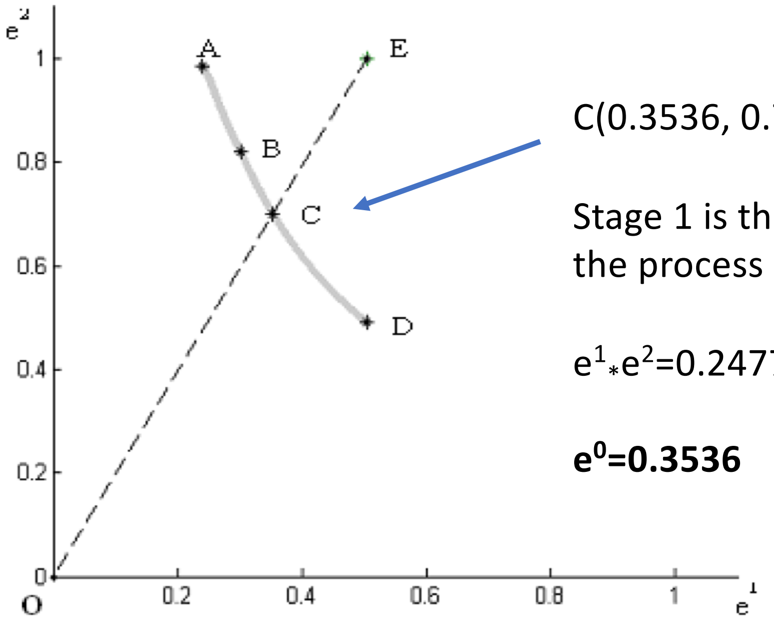
B(**0.3021**, 0.8201)
D(0.5046, **0.4910**)

Joint graphical and MOP representation

$max\ e^o(v,w,u)$
$max\ e^1(v,w,u)$
$max\ e^2(v,w,u)$
s.t.
$e^1(v,w,u) \leq 1$
$e^2(v,w,u) \leq 1$
$v \geq 0, w \geq 0, u \geq 0$

The "weak link" approach

Max [min{(1/E^1)e^1, (1/E^2)e^2}]
to get C



C(0.3536, 0.7007)

Stage 1 is the weak link of the process

$e^1{}_* e^2$=0.2477 as at B and D

e^0=0.3536

Summary

- The independent assessments approach does not take into account the interdependency of the stages
- The leader follower paradigm requires external information and provides extreme efficient scores
- The top-down decomposition approach (multiplicative, additive)
  - is computationally efficient (linear models) but
  - leads to biased and/or non-unique efficiency scores
  - Does not comply with the dominance relation requirement
- The bottom-up approach
  - requires more computational effort (non-linear model in phase I) but
  - leads to unique, unbiased and balanced efficiency assessments
  - Complies with the dominance relation requirement
  - Can provide any other point on the Pareto front by appropriately changing the search direction

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